



**BHARATHIDASANAR MATRIC HIGHER SECONDARY SCHOOL
ARAKONAM**

XII – MATHEMATICS

MATERIAL – 9 & 10 UNIT (6 MARKS & 10 MARKS)

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DISCRETE MATHEMATICS

Ten marks questions:

1. Show that $(\mathbb{Z}, *)$ is an infinite abelian group where $*$ is defined as $a * b = a + b + 2$.

Solution:

(i) **Closure axiom :**

Since a, b and 2 are integers $a + b + 2$ is also an integer.

$$\therefore a * b \in \mathbb{Z} \quad \forall a, b \in \mathbb{Z}$$

Thus closure axiom is true.

(ii) **Associative axiom :**

Let $a, b, c \in G$

$$(a * b) * c = (a + b + 2) * c = (a + b + 2) + c + 2 = a + b + c + 4$$

$$a * (b * c) = a * (b + c + 2) = a + (b + c + 2) + 2 = a + b + c + 4$$

$$\Rightarrow (a * b) * c = a * (b * c)$$

Thus associative axiom is true.

(iii) **Identity axiom :**

Let e be the identity element.

By the definition of e , $a * e = a$

By the definition of $*$, $a * e = a + e + 2$

$$\Rightarrow a + e + 2 = a$$

$$\Rightarrow e = -2$$

$-2 \in \mathbb{Z}$. Thus identity axiom is true.

(iv) **Inverse axiom :**

Let $a \in G$ and a^{-1} be the inverse element of a

By the definition of a^{-1} , $a * a^{-1} = e = -2$

By the definition of $*$, $a * a^{-1} = a + a^{-1} + 2$

$$\Rightarrow a + a^{-1} + 2 = -2$$

$$\Rightarrow a^{-1} = -a - 4$$

Clearly $-a - 4 \in \mathbb{Z}$. \therefore Inverse axiom is true. $\therefore (\mathbb{Z}, *)$ is a group.

(v) **Commutative property :**

Let $a, b \in G$

$a * b = a + b + 2 = b + a + 2 = b * a \therefore *$ is commutative.

$\therefore (Z, *)$ is an abelian group. further, Z is an infinite set. The group is an infinite abelian group.

2. Show that the set G of all matrices of the form $\begin{pmatrix} x & x \\ x & x \end{pmatrix}$, where $x \in R - \{0\}$, is a group under matrix multiplication.

Solution:

Let $G = \left\{ \begin{pmatrix} x & x \\ x & x \end{pmatrix} \mid x \in R - \{0\} \right\}$ we shall show that G is a group under

matrix multiplication.

(i) **Closure axiom :**

$$A = \begin{pmatrix} x & x \\ x & x \end{pmatrix} \in G, B = \begin{pmatrix} y & y \\ y & y \end{pmatrix} \in G$$

$$AB = \begin{pmatrix} 2xy & 2xy \\ 2xy & 2xy \end{pmatrix} \in G (x \neq 0, y \neq 0 \Rightarrow 2xy \neq 0)$$

i.e., G is closed under matrix multiplication.

(ii) Matrix multiplication is always associative.

(iii) Let $E = \begin{pmatrix} e & e \\ e & e \end{pmatrix} \in G$ be such that $AE = A$ for every $A \in G$.

$$AE = A \Rightarrow \begin{pmatrix} x & x \\ x & x \end{pmatrix} \begin{pmatrix} e & e \\ e & e \end{pmatrix} = \begin{pmatrix} x & x \\ x & x \end{pmatrix}$$

$$\begin{pmatrix} 2xe & 2xe \\ 2xe & 2xe \end{pmatrix} = \begin{pmatrix} x & x \\ x & x \end{pmatrix}$$

$$\Rightarrow 2xe = x \Rightarrow e = \frac{1}{2} \quad (x \neq 0)$$

Thus $E = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \in G$ is such that $AE = A$, for every $A \in G$

We can similarly show that $EA = A$ for every $A \in G$.

$\therefore E$ is the identity element in G and hence identity axiom is true.

(iv) Suppose $A^{-1} = \begin{pmatrix} y & y \\ y & y \end{pmatrix} \in G$ is such that $A^{-1}A = E$

$$\text{Then we have } \begin{pmatrix} 2xy & 2xy \\ 2xy & 2xy \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$$

$$\Rightarrow 2xy = 1/2$$

$$y = \frac{1}{4x}$$

$$\therefore A^{-1} = \begin{pmatrix} 1/4x & 1/4x \\ 1/4x & 1/4x \end{pmatrix} \in G \text{ is such that } A^{-1}A = E$$

Similarly we can show that $AA^{-1} = E$. $\therefore A^{-1}$ is the inverse of A .

$\therefore G$ is a group under matrix multiplication.

3. Show that the set $G = \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$ is an infinite abelian group with respect to addition.

Solution:

(i) **Closure axiom :**

Let $x, y \in G$. Then $x = a + b\sqrt{2}$, $y = c + d\sqrt{2}$; $a, b, c, d \in \mathbb{Q}$.

$$x + y = (a + b\sqrt{2}) + (c + d\sqrt{2}) = (a + c) + (b + d)\sqrt{2} \in G,$$

since $(a + c)$ and $(b + d)$ are rational numbers.

$\therefore G$ is closed with respect to addition.

(ii) **Associative axiom :** Since the elements of G are all real numbers, addition

is associative.

(iii) **Identity axiom :**

There exists $0 = 0 + 0\sqrt{2} \in G$ such that for all $x = a + b\sqrt{2} \in G$,

$$x + 0 = (a + b\sqrt{2}) + (0 + 0\sqrt{2})$$

$$= a + b\sqrt{2} = x$$

Similarly, we have $0 + x = x$. $\therefore 0$ is the identity element of G and satisfies the identity axiom.

(iv) **Inverse axiom :**

For each $x = a + b\sqrt{2} \in G$, there exists $-x = (-a) + (-b)\sqrt{2} \in G$ such that $x + (-x) = (a + b\sqrt{2}) + ((-a) + (-b)\sqrt{2})$

$$= (a + (-a)) + (b + (-b))\sqrt{2} = 0$$

Similarly we have $(-x) + x = 0$

$\therefore (-a) + (-b)\sqrt{2}$ is the inverse of $a + b\sqrt{2}$ and satisfies the inverse axiom. $\therefore G$ is a group under addition.

(v) **Commutative axiom :**

$$\begin{aligned} x + y &= (a + c) + (b + d)\sqrt{2} = (c + a) + (d + b)\sqrt{2} \\ &= (c + d\sqrt{2}) + (a + b\sqrt{2}) \end{aligned}$$

$= y + x$, for all $x, y \in G$. \therefore The commutative property is true.

$\therefore (G, +)$ is an abelian group. Since G is infinite, we see that $(G, +)$ is an infinite abelian group.

4. Let G be the set of all rational numbers except 1 and $*$ be defined on G by $a * b = a + b - ab$ for all $a, b \in G$. Show that $(G, *)$ is an infinite abelian group.

Solution: Let $G = \mathbb{Q} - \{1\}$

Let $a, b \in G$. Then a and b are rational numbers and $a \neq 1, b \neq 1$.

(i) **Closure axiom :** Clearly $a * b = a + b - ab$ is a rational number. But to

prove $a * b \in G$, we have to prove that $a * b \neq 1$.

On the contrary, assume that $a * b = 1$ then

$$a + b - ab = 1$$

$$\Rightarrow b - ab = 1 - a$$

$$\Rightarrow b(1 - a) = 1 - a$$

$$\Rightarrow b = 1 \quad (a \neq 1, 1 - a \neq 0)$$

This is impossible, because $b \neq 1$. \therefore Our assumption is wrong.

$\therefore a * b \neq 1$ and hence $a * b \in G$.

∴ Closure axiom is true.

(ii) **Associative axiom :**

$$\begin{aligned} a * (b * c) &= a * (b + c - b c) \\ &= a + (b + c - b c) - a (b + c - b c) \\ &= a + b + c - b c - a b - a c + a b c \\ (a * b) * c &= (a + b - a b) * c \\ &= (a + b - a b) + c - (a + b - a b) c \\ &= a + b + c - a b - a c - b c + a b c \end{aligned}$$

$$\therefore a * (b * c) = (a * b) * c \quad \forall a, b, c \in G$$

∴ Associative axiom is true.

(iii) **Identity axiom :** Let e be the identity element.

By definition of e , $a * e = a$

By definition of $*$, $a * e = a + e - a e$

$$\Rightarrow a + e - a e = a$$

$$\Rightarrow e(1 - a) = 0$$

$$\Rightarrow e = 0 \text{ since } a \neq 1$$

$$e = 0 \in G$$

∴ Identity axiom is satisfied.

(iv) **Inverse axiom :**

Let a^{-1} be the inverse of $a \in G$.

By the definition of inverse, $a * a^{-1} = e = 0$

By the definition of $*$, $a * a^{-1} = a + a^{-1} - a a^{-1}$

$$\Rightarrow a + a^{-1} - a a^{-1} = 0$$

$$\Rightarrow a^{-1} (1 - a) = -a$$

$$\Rightarrow a^{-1} = \frac{-a}{1-a} \in G \text{ since } a \neq 1$$

∴ Inverse axiom is satisfied. ∴ $(G, *)$ is a group

(v) **Commutative axiom :**

For any $a, b \in G$, $a * b = a + b - a b$

$$= b + a - b a$$

$$= b * a$$

∴ $*$ is commutative in G and hence $(G, *)$ is an abelian group. Since G is infinite, $(G, *)$ is an infinite abelian group.

5. Prove that the set of four functions f_1, f_2, f_3, f_4 on the set of nonzero complex numbers $\mathbb{C} - \{0\}$ defined by

$$f_1(z) = z, f_2(z) = -z, f_3(z) = \frac{1}{z}$$

and $f_4(z) = -\frac{1}{z} \forall z \in \mathbb{C} - \{0\}$ forms an abelian group with respect to the composition of functions.

Solution: Let $G = \{f_1, f_2, f_3, f_4\}$

$$(f_1 \circ f_1)(z) = f_1(f_1(z)) = f_1(z)$$

$$\therefore f_1 \circ f_1 = f_1$$

$$f_2 \circ f_1 = f_2, f_3 \circ f_1 = f_3, f_4 \circ f_1 = f_4$$

$$\text{Again } (f_2 \circ f_2)(z) = f_2(f_2(z)) = f_2(-z) = -(-z) = z = f_1(z)$$

$$\therefore f_2 \circ f_2 = f_1$$

$$\text{Similarly } f_2 \circ f_3 = f_4, f_2 \circ f_4 = f_3$$

$$(f_3 \circ f_2)(z) = f_3(f_2(z)) = f_3(-z) = -\frac{1}{-z} = \frac{1}{z} = f_4(z)$$

$$\therefore f_3 \circ f_2 = f_4$$

$$\text{Similarly } f_3 \circ f_3 = f_1, f_3 \circ f_4 = f_2$$

$$(f_4 \circ f_2)$$

$$(f_4 \circ f_2)(z) = f_4(f_2(z)) = f_4(-z) = -\frac{1}{-z} = \frac{1}{z} = f_3(z)$$

$$\therefore f_4 \circ f_2 = f_3$$

$$\text{Similarly } f_4 \circ f_3 = f_2, f_4 \circ f_4 = f_1$$

Using these results we have the composition table as given below :

\circ	f_1	f_2	f_3	f_4
f_1	f_1	f_2	f_3	f_4
f_2	f_2	f_1	f_4	f_3
f_3	f_3	f_4	f_1	f_2
f_4	f_4	f_3	f_2	f_1

. From the table

(i) All the entries of the composition table are the elements of G .

∴ Closure axiom is true.

(ii) Composition of functions is in general associative.

(iii) Clearly f_1 is the identity element of G and satisfies the identity axiom.

(iv) From the table :

Inverse of f_1 is f_1 ; Inverse of f_2 is f_2

Inverse of f_3 is f_3 ; Inverse of f_4 is f_4

Inverse axiom is satisfied. (G, o) is a group.

(v) From the table the commutative property is also true.

∴ (G, o) is an abelian group.

6. Show that $(Z_n, +_n)$ forms group.

Solution:

Let $Z_n = \{[0], [1], [2], \dots [n-1]\}$ be the set of all congruence classes modulo n . and let $[l], [m], \in Z_n$ $0 \leq l, m, < n$

(i) **Closure axiom** : By definition

$$[l] +_n [m] = \begin{cases} [l + m] & \text{if } l + m < n \\ [r] & \text{if } l + m \geq n \text{ where } l + m = q \cdot n + r \text{ } 0 \leq r < n \end{cases}$$

In both the cases, $[l + m] \in Z_n$ and $[r] \in Z_n$

∴ Closure axiom is true.

(ii) Addition modulo n is always associative in the set of congruence classes modulo n .

(iii) The identity element $[0] \in Z_n$ and it satisfies the identity axiom.

(iv) The inverse of $[l] \in Z_n$ is $[n - l]$

Clearly $[n - l] \in Z_n$ and

$$[l] +_n [n - l] = [0]$$

$$[n - l] +_n [l] = [0]$$

∴ The inverse axiom is also true. Hence $(Z_n, +_n)$ is a group

7. Show that $(\mathbb{Z}_7 - \{[0]\}, \cdot_7)$ forms a group.

Solution: Let $G = \{[1], [2], \dots [6]\}$

The Cayley's table is

\cdot_7	[1]	[2]	[3]	[4]	[5]	[6]
[1]	[1]	[2]	[3]	[4]	[5]	[6]
[2]	[2]	[4]	[6]	[1]	[3]	[5]
[3]	[3]	[6]	[2]	[5]	[1]	[4]
[4]	[4]	[1]	[5]	[2]	[6]	[3]
[5]	[5]	[3]	[1]	[6]	[4]	[2]
[6]	[6]	[5]	[4]	[3]	[2]	[1]

From the table :

(i) all the elements of the composition table are the elements of G .

\therefore The closure axiom is true.

(ii) multiplication modulo 7 is always associative.

(iii) the identity element is $[1] \in G$ and satisfies the identity axiom.

(iv) the inverse of $[1]$ is $[1]$; $[2]$ is $[4]$; $[3]$ is $[5]$; $[4]$ is $[2]$; $[5]$ is $[3]$ and

$[6]$ is $[6]$ and it satisfies the inverse axiom.

\therefore The given set forms a group under multiplication modulo 7.

8. Show that the n th roots of unity form an abelian group of finite order with usual multiplication.

Solution: We know that $1, \omega, \omega^2, \dots, \omega^{n-1}$ are the n th roots of unity, where

$$\omega = \text{cis} 2\pi/n$$

$$n. \text{ Let } G = \{1, \omega, \omega^2, \dots, \omega^{n-1}\}$$

(i) **Closure axiom :** Let $\omega^l, \omega^m \in G, 0 \leq l, m \leq (n-1)$

To prove $\omega^l \omega^m = \omega^{l+m} \in G$

Case (i) $l + m < n$

If $l + m < n$ then clearly $\omega^{l+m} \in G$

Case (ii) $l + m \geq n$ By division algorithm,

$l + m = (q \cdot n) + r$ where $0 \leq r < n, q$ is a positive integer.

$$\omega^{l+m} = \omega^{qn+r} = (\omega^n)^q \cdot \omega^r = (1)^q \omega^r = \omega^r \in G \because 0 \leq r < n$$

Closure property is true.

(ii) **Associative axiom** : Multiplication is always associative in the set of complex numbers and hence in G

$$\omega^l \cdot (\omega^p \cdot \omega^m) = \omega^l \cdot \omega^{(p+m)} = \omega^{l+(p+m)} = \omega^{(l+p)+m} = (\omega^{l+p}) \cdot \omega^m \\ = (\omega^l \cdot \omega^p) \cdot \omega^m = \forall \omega^l, \omega^m, \omega^p \in G$$

(iii) **Identity axiom** : The identity element $1 \in G$ and it satisfies

$$1 \cdot \omega^l = \omega^l \cdot 1 = \omega^l \quad \forall \omega^l \in G$$

(iv) **Inverse axiom** :

$$\text{For any } \omega^l \in G, \omega^{n-l} \in G \text{ and } \omega \cdot \omega^{n-l} = \omega^{n-l} \cdot \omega^l = \omega^n = 1$$

Thus inverse axiom is true.

$\therefore (G, \cdot)$ is a group.

(v) **Commutative axiom** :

$$\omega^l \cdot \omega^m = \omega^{l+m} = \omega^{m+l} = \omega^m \cdot \omega^l \quad \forall \omega^l, \omega^m \in G$$

$\therefore (G, \cdot)$ is an abelian group. Since G contains n elements, (G, \cdot) is a finite abelian group of order n .

9. Show that the set G of all positive rationals forms a group under the composition $*$ defined by $a * b = \frac{ab}{3}$ for all $a, b \in G$.

Solution:

(i) closure axiom:

Let $a, b \in G$. Since a, b are positive rational,

ab is positive rational and hence $\frac{ab}{3}$ is also positive rational

$$\frac{ab}{3} \in G \quad \text{i.e., } a * b \in G$$

\therefore Closure axiom is true

(ii) Associative axiom:

Let $a, b, c \in G$

$$(a * b) * c = a * (b * c)$$

$$\frac{ab}{3} * c = a * \frac{bc}{3}$$

$$\frac{abc}{9} = \frac{abc}{9}$$

L.H.S. = R.H.S.

Associative axiom is true

(iii) Identity axiom:

Let e be the identity element .

$$a * e = e * a = a$$

$$a * e = a$$

$$\frac{ae}{3} = a$$

$$e = 3 \in G$$

∴ identity axiom is true

(iv) Inverse axiom:

Let $a \in G$ and a^{-1} is the inverse element of a

$$a * a^{-1} = a^{-1} * a = e$$

$$a * a^{-1} = e$$

$$\frac{aa^{-1}}{3} = 3$$

$$a^{-1} = \frac{9}{a} \in G$$

∴ Inverse axiom is true

*∴ $(G, *)$ is a group*

10. Let G be the set of all rational numbers except -1 and $*$ be defined on G by $a * b = a + b + ab$ for all $a, b \in G$. Show that $(G, *)$ is an infinite abelian group.

Solution: Let $G = Q - \{-1\}$

Let $a, b \in G$. Then a and b are rational numbers and $a \neq -1, b \neq -1$.

(i) **Closure axiom :** Clearly $a * b = a + b + ab$ is a rational number.

But to prove $a * b \in G$, we have to prove that $a * b \neq -1$.

On the contrary, assume that $a * b = 1$ then

$$a + b + ab = -1$$

$$\Rightarrow b + ab = -1 - a$$

$$\Rightarrow b(1 + a) = -(1 + a)$$

$$\Rightarrow b = -1 \quad (\because a \neq -1, 1 + a \neq 0)$$

This is impossible, because $b \neq -1$. \therefore Our assumption is wrong.

$\therefore a * b \neq -1$ and hence $a * b \in G$.

\therefore Closure axiom is true.

(ii) **Associative axiom :**

$$a * (b * c) = a * (b + c + b c)$$

$$= a + (b + c + b c) + a (b + c + b c)$$

$$= a + b + c + b c + a b + a c + a b c$$

$$(a * b) * c = (a + b + a b) * c$$

$$= (a + b + a b) + c + (a + b + a b) c$$

$$= a + b + c + a b + a c + b c + a b c$$

$$\therefore a * (b * c) = (a * b) * c \quad \forall a, b, c \in G$$

\therefore Associative axiom is true.

(iii) **Identity axiom :** Let e be the identity element.

By definition of e , $a * e = a$

By definition of $*$, $a * e = a + e + a e$

$$\Rightarrow a + e + a e = a$$

$$\Rightarrow e(1 + a) = 0$$

$$\Rightarrow e = 0 \text{ since } a \neq -1$$

$$e = 0 \in G$$

\therefore Identity axiom is satisfied.

(iv) **Inverse axiom :**

Let a^{-1} be the inverse of $a \in G$.

By the definition of inverse, $a * a^{-1} = e = 0$

By the definition of $*$, $a * a^{-1} = a + a^{-1} - a a^{-1}$

$$\Rightarrow a + a^{-1} - a a^{-1} = 0$$

$$\Rightarrow a^{-1}(1+a) = -a$$

$$\Rightarrow a^{-1} = \frac{-a}{a-1} \in G \text{ since } a \neq -1$$

\therefore Inverse axiom is satisfied. $\therefore (G, *)$ is a group

(v) Commutative axiom :

For any $a, b \in G$, $a * b = a + b + ab$

$$= b + a + ba$$

$$= b * a$$

$\therefore *$ is commutative in G and hence $(G, *)$ is an abelian group. Since G is infinite, $(G, *)$ is an infinite abelian group.

11. Show that the set $\{[1], [3], [4], [5], [9]\}$ forms an abelian group under multiplication modulo 11.

Let $G = \{[1], [3], [4], [5], [9]\}$

The Cayley's table is

\cdot_{11}	[1]	[3]	[4]	[5]	[9]
[1]	[1]	[3]	[4]	[5]	[9]
[3]	[3]	[9]	[1]	[4]	[5]
[4]	[4]	[1]	[5]	[9]	[3]
[5]	[5]	[4]	[9]	[3]	[1]
[9]	[9]	[5]	[3]	[1]	[4]

From the table :

(i) all the elements of the composition table are the elements of G .

\therefore The closure axiom is true.

(ii) multiplication modulo 11 is always associative.

(iii) the identity element is $[1] \in G$ and satisfies the identity axiom.

(iv) the inverse of $[1]$ is $[1]$; $[3]$ is $[4]$; $[4]$ is $[3]$; $[5]$ is $[9]$ and $[9]$ is $[5]$

and it satisfies the inverse axiom.

\therefore The given set forms a group under multiplication modulo 11.

12. Show that the set G of all matrices of the form $\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$ where $x \in R - \{0\}$, is a group under matrix multiplication.

Solution:

Let $G = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \mid a \in R - \{0\} \right\}$ we shall show that G is a group under matrix multiplication.

(i) **Closure axiom :**

$$A = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \in G, B = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \in G$$

$$AB = \begin{pmatrix} ab & 0 \\ 0 & 0 \end{pmatrix} \in G \quad (a \neq 0, b \neq 0 \Rightarrow ab \neq 0)$$

i.e., G is closed under matrix multiplication.

(ii) Matrix multiplication is always associative.

(iii) Let $E = \begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix} \in G$ be such that $AE = A$ for every $A \in G$.

$$AE = A \Rightarrow \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$$

$$\Rightarrow ae = a \Rightarrow e = 1 \quad (a \neq 0)$$

Thus $E = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in G$ is such that $AE = A$, for every $A \in G$

We can similarly show that $EA = A$ for every $A \in G$.

$\therefore E$ is the identity element in G and hence identity axiom is true.

(iv) Suppose $A^{-1} = \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} \in G$ is such that $A^{-1}A = E$

Then we have $\begin{pmatrix} xa & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

$\Rightarrow xa = 1$

$x = \frac{1}{a}$

$\therefore A^{-1} = \begin{pmatrix} 1/a & 0 \\ 0 & 0 \end{pmatrix} \in G$ is such that $A^{-1}A = E$

Similarly we can show that $AA^{-1} = E$. $\therefore A^{-1}$ is the inverse of A .

$\therefore G$ is a group under matrix multiplication.

(v) Commutative axiom:

$A, B \in G$

$AB = \begin{pmatrix} ab & 0 \\ 0 & 0 \end{pmatrix}$

$= \begin{pmatrix} ba & 0 \\ 0 & 0 \end{pmatrix}$

$= BA \quad (a, b \in R \Rightarrow ab = ba)$

\therefore The group is an abelian group under matrix multiplication

13. Show that the set $G = \{2^n / n \in \mathbb{Z}\}$ is an abelian group under multiplication.

Solution:

$G = \{2^n / n \in \mathbb{Z}\}$

(i) closure axiom :

Let $x = 2^r \quad y = 2^s \in G$ where $r, s \in \mathbb{Z}$

$xy = 2^r \cdot 2^s = 2^{r+s} \in G \quad (r, s \in \mathbb{Z} \Rightarrow r + s \in \mathbb{Z})$

\therefore The closure axiom is true.

(ii) Associative axiom:

$$x = 2^r, y = 2^s, z = 2^t \in G \text{ where } r, s, t \in Z$$

$$\begin{aligned}(x \cdot y) \cdot z &= (2^r \cdot 2^s) \cdot 2^t = 2^{r+s} \cdot 2^t = 2^{(r+s)+t} \\ &= 2^{r+(s+t)} = 2^r \cdot (2^s \cdot 2^t) = x \cdot (y \cdot z)\end{aligned}$$

\therefore Associative axiom is true

(iii) Identity axiom:

For every $x = 2^r \in G$, there exist $1 = 2^0 \in G$

$$\text{Such that } x \cdot 1 = 2^r \cdot 1 = 2^r = x$$

$$\text{Similarly } 1 \cdot x = 1 \cdot 2^r = 2^r = x$$

\therefore 1 is the identity element and hence the identity axiom is true

(iv) Inverse axiom:

For every $x = 2^r \in G$, there exist

$$x^{-1} = 2^{-r} \in G \text{ such that } x \cdot x^{-1} = 2^r \cdot 2^{-r} = 2^{r+(-r)} = 2^0 = 1$$

$$\text{Similarly } x^{-1} \cdot x = 2^{-r} \cdot 2^r = 2^0 = 1$$

$$\therefore \text{Inverse of } 2^r \text{ is } 2^{-r} \in G =$$

$>$ inverse axiom is true and

therefore G is a group

(v) Commutative axiom:

$$x = 2^r, y = 2^s \in G$$

$$x \cdot y = 2^r \cdot 2^s = 2^{r+s} = 2^{s+r} = 2^s \cdot 2^r = y \cdot x$$

commutative axiom is true

(G, \cdot) is an abelian group

14. Show that the set M of complex numbers z with the condition $|z| = 1$ forms a group with respect to the operation of multiplication of complex numbers.

Solution:

$$M = \{ z \in \mathbb{C} \mid |z| = 1 \}$$

(i) Closure axiom:

$$\text{Let } z_1, z_2 \in M$$

$$|z_1 z_2| = |z_1| |z_2| = 1 \cdot 1 = 1 \Rightarrow z_1 z_2 \in M$$

\therefore Closure axiom is true

(ii) Associative axiom:

The multiplication of complex number is always associative

(iii) Identity axiom:

$$|1| = 1 \in \mathbb{C} \text{ such that}$$

$$z \cdot 1 = 1 \cdot z = z$$

$\therefore 1$ is the identity element

\therefore Identity axiom is true

(iv) Inverse axiom:

$$\text{Let } z \in M \text{ . } |z| = 1$$

$$\left| \frac{1}{z} \right| = \frac{1}{|z|} = \frac{1}{1} = 1 \Rightarrow \frac{1}{z} \in M \text{ and } z \cdot \frac{1}{z} = \frac{1}{z} \cdot z = 1$$

$\therefore \frac{1}{z}$ is the inverse of $z \in M$

\therefore Inverse axiom is true

∴ M is a group with respect to the multiplication of complex number

15. Show that

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}, \begin{pmatrix} \omega^2 & 0 \\ 0 & \omega \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \omega^2 \\ \omega & 0 \end{pmatrix}, \begin{pmatrix} 0 & \omega \\ \omega^2 & 0 \end{pmatrix} \right\}$$

where $\omega^3 = 1$, $\omega \neq 1$ form a group with respect to matrix multiplication.

Solution:

$$\text{Let } I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, A = \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}, B = \begin{pmatrix} \omega^2 & 0 \\ 0 & \omega \end{pmatrix}, C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$D = \begin{pmatrix} 0 & \omega^2 \\ \omega & 0 \end{pmatrix}, E = \begin{pmatrix} 0 & \omega \\ \omega^2 & 0 \end{pmatrix}$$

Let $G = \{I, A, B, C, D, E\}$

By computing the products of these matrices, taken in pairs, we can form the multiplication table (Cayley's table) as given below:

.	I	A	B	C	D	E
I	I	A	B	C	D	E
A	A	B	I	E	C	D
B	B	I	A	D	E	C
C	C	D	E	I	A	B

D D E C B I A

E E C D A B I

(i) All the entries in the multiplication table are members of G. So G is closed under multiplication of matrices.

∴ Closure axiom is true

(ii) Since matrix multiplication is associative in general, we see that '·' is associative

(iii) From the table, it is clear that, I is the identity element in G.

(iv) $I \cdot I = I \Rightarrow I$ is the inverse of I

A. $B \cdot B \cdot A = I \Rightarrow A$ and B are inverses of each other.

C. $C \cdot C = I \Rightarrow C$ is the inverse of C

D. $D \cdot D = I \Rightarrow D$ is the inverse of D

E. $E \cdot E = I \Rightarrow E$ is the inverse of E

∴ G is a group with respect to matrix multiplication.

Six marks questions:

1. Construct the truth table for $(p \wedge q) \vee (\sim(p \wedge q))$

Solution:

Truth table for $(p \wedge q) \vee (\sim(p \wedge q))$

p	q	$P \wedge q$	$\sim(p \wedge q)$	$(p \wedge q) \vee (\sim(p \wedge q))$
T	T	T	F	T
T	F	F	T	T
F	T	F	T	T
F	F	F	T	T

2. Construct the truth table for $(p \vee q) \vee r$

Solution:

Truth table for $(p \vee q) \vee r$

p	q	r	$p \vee q$	$(p \vee q) \vee r$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	T	T
F	T	T	T	T
F	T	F	T	T
F	F	T	F	T
F	F	F	F	F

3. Construct the truth table for $(p \wedge q) \vee r$

Solution:

Truth table for $(p \wedge q) \vee r$

p	q	r	$P \wedge q$	$(p \wedge q) \vee r$
T	T	T	T	T
T	T	F	T	T
T	F	T	F	T
T	F	F	F	F
F	T	T	F	T
F	T	F	F	F
F	F	T	F	T
F	F	F	F	F

4. Use the truth table to establish which of the following statements are tautologies and which are contradictions.

(i) $((\sim p) \wedge q) \wedge p$ (ii) $(p \vee q) \vee (\sim(p \vee q))$ (iii) $(p \wedge (\sim q)) \vee ((\sim p) \vee q)$

(iv) $q \vee (p \vee (\sim q))$ (v) $(p \wedge (\sim p)) \wedge ((\sim q) \wedge p)$

Solution:

(i) Truth table for $((\sim p) \wedge q) \wedge p$

p	q	$\sim p$	$(\sim p) \wedge q$	$((\sim p) \wedge q) \wedge p$
T	T	F	F	F
T	F	F	F	F
F	T	T	T	F
F	F	T	F	F

The last column contains only F

$\therefore ((\sim p) \wedge q) \wedge p$ is a contradiction

(ii) Truth table for $(p \vee q) \vee (\sim(p \vee q))$

p	q	$p \vee q$	$\sim(p \vee q)$	$(p \vee q) \vee (\sim(p \vee q))$
T	T	T	F	T
T	F	T	F	T
F	T	T	F	T
F	F	F	T	T

The last column contains only T

$\therefore (p \vee q) \vee (\sim(p \vee q))$ is a tautology

(iii) Truth table for $(p \wedge (\sim q)) \vee ((\sim p) \vee q)$

p	q	$\sim p$	$\sim q$	$p \wedge (\sim q)$	$(\sim p) \vee q$	$(p \wedge (\sim q)) \vee ((\sim p) \vee q)$
T	T	F	F	F	T	T
T	F	F	T	T	F	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

The last column contains only T

$\therefore (p \wedge (\sim q)) \vee ((\sim p) \vee q)$ is a tautology

(iv) Truth table for $q \vee (p \vee (\sim q))$

p	q	$\sim q$	$P \vee (\sim q)$	$q \vee (p \vee (\sim q))$
T	T	F	T	T
T	F	T	T	T
F	T	F	F	T
F	F	T	T	T

The last columns contains only T

$\therefore q \vee (p \vee (\sim q))$ is a tautology

(v) Truth table for $(p \wedge (\sim p)) \wedge ((\sim q) \wedge p)$

p	q	$\sim p$	$\sim q$	$p \vee (\sim q)$	$(\sim q) \wedge p$	$(p \wedge (\sim p)) \wedge ((\sim q) \wedge p)$
T	T	F	F	F	F	F
T	F	F	T	F	T	F
F	T	T	F	F	F	F
F	F	T	T	F	F	F

The last columns contain only F

$\therefore (p \wedge (\sim p)) \wedge ((\sim q) \wedge p)$ is a contradiction

5. Show that $p \rightarrow q \equiv (\sim p) \vee q$

Solution:

Truth table for $p \rightarrow q$

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	F

Truth table for $(\sim p) \vee q$

p	q	$\sim p$	$(\sim p) \vee q$
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

The last columns in the truth tables of $p \rightarrow q$ and $(\sim p) \vee q$ are identical

$$\therefore p \rightarrow q \equiv (\sim p) \vee q$$

6. Show that $p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$

Solution:

Truth table for $p \leftrightarrow q$

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Truth table for $(p \rightarrow q) \wedge (q \rightarrow p)$

p	q	$p \rightarrow q$	$q \rightarrow p$	$(p \rightarrow q) \wedge (q \rightarrow p)$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F

F F T T T

Both the tables have identical last columns.

$$\therefore p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$$

7. Show that $p \leftrightarrow q \equiv ((\sim p) \vee q) \wedge ((\sim q) \vee p)$

Solution:

Truth table for $p \leftrightarrow q$

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Truth table for $((\sim p) \vee q) \wedge ((\sim q) \vee p)$

p	q	$\sim p$	$\sim q$	$(\sim p) \vee q$	$(\sim q) \vee p$	$((\sim p) \vee q) \wedge ((\sim q) \vee p)$
T	T	F	F	T	T	T
T	F	F	T	F	T	F

F	T	T	F	T	F	F
F	F	T	T	T	T	T

Both the truth tables have last columns as identical

$$\therefore p \leftrightarrow q \equiv ((\sim p) \vee q) \wedge ((\sim q) \vee p)$$

8. Show that $\sim(p \wedge q) \equiv ((\sim p) \vee (\sim q))$

Solution:

Truth table for $\sim(p \wedge q)$

p	q	$p \wedge q$	$\sim(p \wedge q)$
T	T	T	F
T	F	F	T
F	T	F	T
F	F	F	T

Truth table for $(\sim p) \vee (\sim q)$

p	q	$\sim p$	$\sim q$	$(\sim p) \vee (\sim q)$
T	T	F	F	F
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

Both the table have identical last columns.

$$\therefore \sim(p \wedge q) \equiv ((\sim p) \vee (\sim q))$$

9. Show that $p \rightarrow q$ and $q \rightarrow p$ are not equivalent.

Solution:

Truth table for $p \rightarrow q$ and $q \rightarrow p$

p	q	$p \rightarrow q$	$q \rightarrow p$
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

The columns corresponding to $p \rightarrow q$ and $q \rightarrow p$ are not identical

$\therefore p \rightarrow q$ is not equivalent to $q \rightarrow p$

10. Show that $(p \wedge q) \rightarrow (p \vee q)$ is a tautology

Solution:

Truth table for $(p \wedge q) \rightarrow (p \vee q)$

p	q	$p \wedge q$	$p \vee q$	$(p \wedge q) \rightarrow (p \vee q)$
T	T	T	T	T
T	F	F	T	T
F	T	F	T	T
F	F	F	F	T

The last column contains only T

$\therefore (p \wedge q) \rightarrow (p \vee q)$ is a tautology

11. Construct truth table for $(p \vee q) \wedge (\sim q)$

Solution:

Truth table for $(p \vee q) \wedge (\sim q)$

p	q	$p \vee q$	$\sim q$	$(p \vee q) \wedge (\sim q)$
T	T	T	F	F
T	F	T	T	T
F	T	T	F	F
F	F	F	T	F

12. Construct truth table for $\sim((\sim p) \wedge (\sim q))$

Solution:

Truth table for $\sim((\sim p) \wedge (\sim q))$

p	q	$\sim p$	$\sim q$	$(\sim p) \wedge (\sim q)$	$\sim((\sim p) \wedge (\sim q))$
T	T	F	F	F	T
T	F	F	T	F	T
F	T	T	F	F	T
F	F	T	T	T	F

13. Show that $((\sim p) \vee (\sim q)) \vee p$ is a tautology

Solution:

Truth table for $((\sim p) \vee (\sim q)) \vee p$

p	q	$\sim p$	$\sim q$	$(\sim p) \vee (\sim q)$	$((\sim p) \vee (\sim q)) \vee p$
T	T	F	F	F	T
T	F	F	T	T	T
F	T	T	F	T	T
F	F	T	T	T	T

The last columns contains only T

$\therefore ((\sim p) \vee (\sim q)) \vee p$ is a tautology

14. Construct truth table for $(p \wedge q) \vee (\sim r)$

Solution:

Truth table for $(p \wedge q) \vee (\sim r)$

p	q	r	$p \wedge q$	$\sim r$	$(p \wedge q) \vee (\sim r)$
T	T	T	T	F	T
T	T	F	T	T	T
T	F	T	F	F	F
T	F	F	F	T	T
F	T	T	F	F	F
F	T	F	F	T	T
F	F	T	F	F	F
F	F	F	F	T	T

15. Construct truth table for $(p \vee q) \wedge r$

Solution:

Truth table for $(p \vee q) \wedge r$

p	q	r	$p \vee q$	$(p \vee q) \wedge r$
T	T	T	T	T
T	T	F	T	F
T	F	T	F	T
T	F	F	F	F
F	T	T	F	T
F	T	F	F	F
F	F	T	F	F
F	F	F	F	F

16. Show that $\sim(p \vee q) \equiv (\sim p) \wedge (\sim q)$

Solution:

Truth table for $\sim(p \vee q)$

p	q	$p \vee q$	$\sim(p \vee q)$
T	T	T	F
T	F	T	F
F	T	T	F
F	F	F	T

Truth table for $(\sim p) \wedge (\sim q)$

p	q	$\sim p$	$\sim q$	$(\sim p) \wedge (\sim q)$
T	T	F	F	F
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

The last columns are identical

$$\therefore \sim(p \vee q) \equiv (\sim p) \wedge (\sim q)$$

17. Show that $((\sim q) \wedge p) \wedge q$ is a contradiction

Solution:

Truth table for $((\sim q) \wedge p) \wedge q$

p	q	$\sim q$	$(\sim q) \wedge p$	$((\sim q) \wedge p) \wedge q$
T	T	F	F	F
T	F	T	T	F
F	T	F	F	F
F	F	T	F	F

The last column contains only F

$\therefore ((\sim q) \wedge p) \wedge q$ is a contradiction

18. Use the truth table to determine whether the statement

$((\sim p) \vee q) \vee (p \wedge (\sim q))$ is a tautology

Solution:

Truth table for $((\sim p) \vee q) \vee (p \wedge (\sim q))$

p	q	$\sim p$	$\sim q$	$(\sim p) \vee q$	$p \wedge (\sim q)$	$((\sim p) \vee q) \vee (p \wedge (\sim q))$
T	T	F	F	T	F	T
T	F	F	T	F	T	T
F	T	T	F	T	F	T
F	F	T	T	T	F	T

The last column contains only T

$\therefore ((\sim p) \vee q) \vee (p \wedge (\sim q))$ is a tautology

18. Show that $(R - \{0\}, \cdot)$ is an infinite abelian group. Here ' \cdot ' denotes usual multiplication.

Solution:

(i) **Closure axiom** : Since product of two non-zero real numbers is again a non-zero a real number.

i.e., $\forall a, b \in R, a \cdot b \in R$.

(ii) **Associative axiom** : Multiplication is always associative in $R - \{0\}$

i.e., $a \cdot (b \cdot c) = (a \cdot b) \cdot c \quad \forall a, b, c \in R - \{0\}$

\therefore associative axiom is true.

(iii) **Identity axiom** : The identity element is $1 \in R - \{0\}$ under multiplication and

$$1 \cdot a = a \cdot 1 = a, \forall a \in R - \{0\}$$

\therefore Identity axiom is true.

(iv) **Inverse axiom** : $\forall a \in R - \{0\}$, such that

$$a \cdot \frac{1}{a} = \frac{1}{a} \cdot a = 1$$

\therefore Inverse axiom is true. $\therefore (R - \{0\}, \cdot)$ is a group.

(v) $\forall a, b \in R - \{0\}, a \cdot b = b \cdot a$

\therefore Commutative property is true. $\therefore (R - \{0\}, \cdot)$ is an abelian group

(vi) Further $R - \{0\}$ is an infinite set, $(R - \{0\}, \cdot)$ is an infinite abelian group.

19. Prove that $(\mathbb{Z}, +)$ is an infinite abelian group.

Solution:

(i) **Closure axiom** : We know that sum of two integers is again an integer.

(ii) **Associative axiom** : Addition is always associative in \mathbb{Z}

$$\text{i.e., } \forall a, b, c \in \mathbb{Z}, (a + b) + c = a + (b + c)$$

(iii) **Identity axiom** : The identity element $0 \in \mathbb{Z}$ and it satisfies

$$0 + a = a + 0 = a, \forall a \in \mathbb{Z}$$

Identity axiom is true.

(iv) **Inverse axiom** : For every $a \in \mathbb{Z}$, \exists an element $-a \in \mathbb{Z}$ such that $-a + a = a + (-a) = 0$

\therefore Inverse axiom is true. $\therefore (\mathbb{Z}, +)$ is a group.

(v) $\forall a, b \in \mathbb{Z}, a + b = b + a$

\therefore addition is commutative. $\therefore (\mathbb{Z}, +)$ is an abelian group.

(vi) Since \mathbb{Z} is an infinite set $(\mathbb{Z}, +)$ is infinite abelian group.

20. Show that the cube roots of unity forms a finite abelian group under multiplication.

Solution:

$$\text{Let } G = \{1, \omega, \omega^2\}.$$

The cayley's table is

.	1	ω	ω^2
1	1	ω	ω^2
ω	ω	ω^2	1
ω^2	ω^2	1	ω

From the table, we see that,

(i) all the entries in the table are members of G .
So, the closure property is true.

(ii) multiplication is always associative.

(iii) the identity element is 1 and it satisfies the identity axiom.

(iv) The inverse of 1 is 1

The inverse of ω is ω^2

the inverse of ω^2 is ω

and it satisfies the inverse axiom also

$\therefore (G, \cdot)$ is a group.

(v) the commutative property is also true.

$\therefore (G, \cdot)$ is an abelian group.

(vi) Since G is a finite set, (G, \cdot) is a finite abelian group

21. Prove that the set of all 4th roots of unity forms an abelian group under multiplication.

Solution: We know that the fourth roots of unity are $1, i, -1, -i$.

Let $G = \{1, i, -1, -i\}$. The Cayley's table is

.	1	-1	i	-i
1	1	-1	i	-i
i	i	-i	-1	1
-1	-1	1	-i	i
-i	-i	i	1	-1

From the table,

(i) the closure axiom is true.

(ii) multiplication is always associative in C and hence in G .

(iii) the identity element is $1 \in G$ and it satisfies the identity axiom.

(iv) the inverse of 1 is 1 ; i is $-i$; -1 is -1 ; and $-i$ is i . Further it satisfies the inverse axiom. hence $(G, .)$ is a group.

(v) From the table, the commutative property is also true.

$\therefore (G, .)$ is an abelian group.

22. Prove that $(\mathbb{C}, +)$ is an infinite abelian group.

Solution:

(i) **Closure axiom** : Sum of two complex numbers is always a complex number.

$$\text{i.e., } z_1, z_2 \in \mathbb{C} \Rightarrow z_1 + z_2 \in \mathbb{C}$$

Closure axiom is true.

(ii) **Associative axiom** : Addition is always associative in \mathbb{C}

$$\text{i.e., } (z_1 + z_2) + z_3 = z_1 + (z_2 + z_3) \quad \forall z_1, z_2, z_3 \in \mathbb{C}$$

\therefore Associative axiom is true.

(iii) **Identity axiom** :

The identity element $o = o + i0 \in \mathbb{C}$ and $o + z = z + o = z \quad \forall z \in \mathbb{C}$

\therefore Identity axiom is true.

(iv) **Inverse axiom** : For every $z \in \mathbb{C}$ there exists a unique $-z \in \mathbb{C}$ such that

$$z + (-z) = -z + z = 0. \text{ Inverse is true. } \therefore (\mathbb{C}, +) \text{ is a group.}$$

(v) **Commutative property** :

$$\forall z_1, z_2 \in \mathbb{C}, z_1 + z_2 = z_2 + z_1$$

\therefore the commutative property is true. Hence $(\mathbb{C}, +)$ is an abelian group.

Since \mathbb{C} is an infinite set $(\mathbb{C}, +)$ is an infinite abelian group.

23. Show that the set of all non-zero complex numbers is an abelian group under the usual multiplication of complex numbers.

Solution:

(i) **Closure axiom** : Let $G = \mathbb{C} - \{0\}$ Product of two non-zero complex numbers is again a non-zero complex number.

\therefore Closure axiom is true.

(ii) **Associative axiom** :

Multiplication is always associative.

\therefore Associative property is true.

(iii) **Identity axiom** :

$$1 = 1 + i0 \in G, 1 \text{ is the identity element and } 1 \cdot z = z \cdot 1 = z \quad \forall z \in G.$$

\therefore Identity axiom is true.

(iv) **Inverse axiom :**

Let $z = x + iy \in G$. Here $z \neq 0 \Rightarrow x$ and y are not both zero.

$\therefore x^2 + y^2 \neq 0$

$$\frac{1}{z} = \frac{1}{x+iy} = \frac{x-iy}{(x+iy)(x+iy)} = \frac{x-iy}{x^2+y^2} = \frac{x}{x^2+y^2} + i\left(\frac{-y}{x^2+y^2}\right) \in G$$

Further $z \cdot \frac{1}{z} = \frac{1}{z} \cdot z = 1 \therefore z$ has the inverse $\frac{1}{z} \in G$.

Thus inverse axiom is satisfied. $\therefore (G, \cdot)$ is a group.

24. Show that the set of all 2×2 non-singular matrices forms a non-abelian infinite group under matrix multiplication, (where the entries belong to R).

Solution:

Let G be the set of all 2×2 non-singular matrices, where the entries belong to R .

(i) **Closure axiom :** Since product of two non-singular matrices is again non-singular and the order is 2×2 , the closure axiom is satisfied.

i.e., $A, B \in G \Rightarrow AB \in G$.

(ii) **Associative axiom :** Matrix multiplication is always associative and hence

associative axiom is true. i.e., $A(BC) = (AB)C \quad \forall A, B, C \in G$.

(iii) **Identity axiom :** The identity element is $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in G$ and it satisfies

the identity property.

(iv) **Inverse axiom :** the inverse of $A \in G$, exists i.e. A^{-1} exists and is of order

2×2 and $AA^{-1} = A^{-1}A = I$. Thus the inverse axiom is satisfied. Hence the set of all 2×2 non-singular matrices forms a group under matrix multiplication. Further, matrix multiplication is non-commutative (in

general) and the set contain infinitely many elements. The group is an infinite non-abelian group.

25. Show that the set of four matrices $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ form an abelian, under multiplication of matrices.

Solution:

$$\text{Let } I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad C = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$G = \{I, A, B, C\}$$

By computing the products of these matrices, taken in pairs, we can form

the multiplication table as given below :

$$\cdot \quad I \quad A \quad B \quad C$$

$$I \quad I \quad A \quad B \quad C$$

$$A \quad A \quad I \quad C \quad B$$

$$B \quad B \quad C \quad I \quad A$$

$$C \quad C \quad B \quad A \quad I$$

(i) All the entries in the multiplication tables are members of G . So, G is closed under \cdot . \therefore Closure axiom is true.

(ii) Matrix multiplication is always associative

(iii) Since the row headed by I coincides with the top row and the column

headed by I coincides with the extreme left column, I is the identity element in G .

(iv) $I \cdot I = I \Rightarrow I$ is the inverse of I

$A \cdot A = I \Rightarrow A$ is the inverse of A

$B \cdot B = I \Rightarrow B$ is the inverse of B

C. $C = I \Rightarrow C$ is the inverse of C

From the table it is clear that \cdot is commutative. $\therefore G$ is an abelian group under matrix multiplication.

26. State and prove cancellation laws on groups

Solution:

Let G be a group. Then for all $a, b, c \in G$,

(i) $a * b = a * c \Rightarrow b = c$ (Left Cancellation Law)

(ii) $b * a = c * a \Rightarrow b = c$ (Right Cancellation Law)

Proof : (i) $a * b = a * c \Rightarrow a^{-1} * (a * b) = a^{-1} * (a * c)$

$$\Rightarrow (a^{-1} * a) * b = (a^{-1} * a) * c$$

$$\Rightarrow e * b = e * c$$

$$\Rightarrow b = c$$

(ii) $b * a = c * a \Rightarrow (b * a) * a^{-1} = (c * a) * a^{-1}$

$$\Rightarrow b * (a * a^{-1}) = c * (a * a^{-1})$$

$$\Rightarrow b * e = c * e$$

$$\Rightarrow b = c$$

27. State and prove reversal law on inverses of group

Solution:

Let G be a group $a, b \in G$. Then $(a * b)^{-1} = b^{-1} * a^{-1}$.

Proof : It is enough to prove $b^{-1} * a^{-1}$ is the inverse of $(a * b)$

\therefore To prove (i) $(a * b) * (b^{-1} * a^{-1}) = e$

(ii) $(b^{-1} * a^{-1}) * (a * b) = e$

(i) $(a * b) * (b^{-1} * a^{-1}) = a * (b * b^{-1}) * a^{-1}$

$$= a * (e) * a^{-1}$$

$$= a * a^{-1} = e$$

$$\begin{aligned}
 \text{ii) } & (b^{-1} * a^{-1}) * (a * b) = b^{-1} * (a^{-1} * a) * b \\
 & = b^{-1} * (e) * b \\
 & = b^{-1} * b = e
 \end{aligned}$$

$\therefore b^{-1} * a^{-1}$ is the inverse of $a * b$ i.e., $(a * b)^{-1} = b^{-1} * a^{-1}$

For two subdivisions – Each 3 Mark

1. Construct the truth table for $p \vee (\sim q)$

Solution:

Truth table for $p \vee (\sim q)$

p	q	$\sim q$	$p \vee (\sim q)$
T	T	F	T
T	F	T	T
F	T	F	F
F	F	T	T

2. Construct the truth table for $(\sim p) \wedge (\sim q)$

Solution:

Truth table for $(\sim p) \wedge (\sim q)$

p	q	$\sim p$	$\sim q$	$(\sim p) \wedge (\sim q)$
---	---	----------	----------	----------------------------

T	T	F	F	F
T	F	F	T	F
F	F	T	F	F
F	F	T	T	T

3. Construct the truth table for $\sim(p \vee q)$

Solution

Truth table *for* $\sim(p \vee q)$

p	q	$p \vee q$	$\sim(p \vee q)$
T	T	T	F
T	F	T	F
F	T	T	F
F	F	F	T

4. Construct the truth table for $(p \vee q) \vee (\sim p)$

Solution:

Truth table *for* $(p \vee q) \vee (\sim p)$

p	q	$p \vee q$	$\sim p$	$(p \vee q) \vee (\sim p)$
-----	-----	------------	----------	----------------------------

T	T	T	F	T
---	---	---	---	---

T	F	T	F	T
---	---	---	---	---

F	T	T	T	T
---	---	---	---	---

F	F	F	T	T
---	---	---	---	---

5. Construct the truth table for $(p \wedge q) \vee (\sim q)$

Solution:

Truth table for $(p \wedge q) \vee (\sim q)$

p	q	$p \wedge q$	$\sim q$	$(p \wedge q) \vee (\sim q)$
-----	-----	--------------	----------	------------------------------

T	T	T	F	T
---	---	---	---	---

T	F	F	T	T
---	---	---	---	---

F	T	F	F	F
---	---	---	---	---

F	F	F	T	T
---	---	---	---	---

6. Construct the truth table for $\sim(p \vee (\sim q))$

Solution:

Truth table for $\sim(p \vee (\sim q))$

p	q	$\sim q$	$p \vee (\sim q)$	$\sim(p \vee (\sim q))$
T	T	F	T	F
T	F	T	T	F
F	T	F	F	T
F	F	T	T	F

7. Construct the truth table for $(p \wedge q) \wedge (\sim q)$

Solution:

Truth table for $(p \wedge q) \wedge (\sim q)$

p	q	$p \wedge q$	$\sim q$	$(p \wedge q) \wedge (\sim q)$
T	T	T	F	F
T	F	F	T	F
F	T	F	F	F
F	F	F	T	F

8. Construct the truth table for $p \vee (\sim q)$

Solution:

Truth table for $p \vee (\sim q)$

p	q	$\sim q$	$p \vee (\sim q)$
T	T	F	T
T	F	T	T
F	T	F	F
F	F	T	T

9. Construct the truth table for $((\sim p) \vee (\sim q))$

Solution:

Truth table for $((\sim p) \vee (\sim q))$

p	q	$\sim p$	$\sim q$	$((\sim p) \vee (\sim q))$
T	T	F	F	F
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

10. Construct the truth table for $\sim((\sim p)^{\wedge}q)$

Solution:

Truth table for $\sim((\sim p)^{\wedge}q)$

p	q	$\sim p$	$(\sim p)^{\wedge}q$	$\sim((\sim p)^{\wedge}q)$
T	T	F	F	T
T	F	F	F	T
F	T	T	T	F
F	F	T	F	T

11. Construct the truth table for $(\sim p)^{\wedge}(\sim q)$

Solution:

Truth table for $(\sim p)^{\wedge}(\sim q)$

p	q	$\sim p$	$\sim q$	$(\sim p)^{\wedge}(\sim q)$
T	T	F	F	F
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

12. Construct the truth table for $\sim(p \vee q)$

Solution:

Truth table for $\sim(p \vee q)$

p	q	$p \vee q$	$\sim(p \vee q)$
---	---	------------	------------------

T	T	T	F
---	---	---	---

T	F	T	F
---	---	---	---

F	T	T	F
---	---	---	---

F	F	F	T
---	---	---	---

13. Construct the truth table for $(p \vee q) \vee (\sim p)$

Solution:

Truth table for $(p \vee q) \vee (\sim p)$

p	q	$\sim p$	$p \vee q$	$(p \vee q) \vee (\sim p)$
T	T	F	T	T
T	F	F	T	T
F	T	T	T	T
F	F	T	F	T

14. Construct the truth table for $(p \wedge q) \vee (\sim q)$

Solution:

Truth table for $(p \wedge q) \vee (\sim q)$

p	q	$\sim q$	$p \wedge q$	$(p \wedge q) \vee (\sim q)$
T	T	F	T	F
T	F	T	F	F
F	T	F	F	F
F	F	T	F	F

15. Construct the truth table for $\sim(p \vee (\sim q))$

Solution: Truth table for $\sim(p \vee (\sim q))$

p	q	$\sim q$	$(p \vee (\sim q))$	$\sim(p \vee (\sim q))$
T	T	F	T	F
T	F	T	T	F
F	T	F	F	T
F	F	T	T	F

16. Prove that identity element of a group is unique

Proof:

Let G be a group. If possible let e_1 and e_2 be identity elements in G .

Treating e_1 as an identity element we have $e_1 * e_2 = e_2 \cdots (1)$

Treating e_2 as an identity element, we have $e_1 * e_2 = e_1 \cdots (2)$

From (1) and (2), $e_1 = e_2$

\therefore Identity element of a group is unique.

17. Prove that inverse element of an element of a group is unique

Proof:

Let G be a group and let $a \in G$.

If possible, let a_1 and a_2 be two inverses of a .

Treating a_1 as an inverse of 'a' we have $a * a_1 = a_1 * a = e$.

Treating a_2 as an inverse of 'a', we have $a * a_2 = a_2 * a = e$

Now $a_1 = a_1 * e = a_1 * (a * a_2) = (a_1 * a) * a_2 = e * a_2 = a_2$

\Rightarrow Inverse of an element is unique.

18. Show that $(a^{-1})^{-1} = a \forall a \in G$, a group

Proof:

We know that $a^{-1} \in G$ and hence $(a^{-1})^{-1} \in G$. Clearly $a * a^{-1} = a^{-1} * a = e$

$a^{-1} * (a^{-1})^{-1} = (a^{-1})^{-1} * a^{-1} = e$

$\Rightarrow a * a^{-1} = (a^{-1})^{-1} * a^{-1}$

$\Rightarrow a = (a^{-1})^{-1}$ (by Right Cancellation Law)

PROBABILITY DISTRIBUTIONS

Ten mark questions:

1. The probability density function of random variable x is

$$f(x) = \begin{cases} kx^{\alpha-1}e^{-\beta x^\alpha}, & x, \alpha, \beta > 0 \\ 0 & , \text{else where} \end{cases}$$

(i)k (ii) P(X > 10)

Solution:

Since f(x) is a probability density function

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

$$k \int_0^{\infty} x^{\alpha-1} e^{-\beta x^\alpha} dx = 1$$

$$\text{Put } t = \beta x^\alpha$$

$$dt = \alpha\beta x^{\alpha-1} dx$$

x	0	∞
t	0	∞

$$\int_0^{\infty} \frac{1}{\alpha\beta} e^{-t} dt = 1$$

$$\frac{k}{\alpha\beta} \left[\frac{e^{-t}}{-1} \right]_0^{\infty} = 1$$

$$\Rightarrow -\frac{k}{\alpha\beta} [0-1]=1$$

$$k = \alpha\beta$$

$$(ii) P(X > 10) = \int_{10}^{\infty} f(x) = \int_{10}^{\infty} \alpha\beta x^{\alpha-1} e^{-\beta x^{\alpha}} dx$$

$$\text{Put } t = \beta x^{\alpha}$$

$$dt = \alpha\beta x^{\alpha-1} dx$$

x	10	∞
t	$\beta(10)^{\alpha}$	∞

$$= \int_{\beta(10)^{\alpha}}^{\infty} e^{-t} dt$$

$$= \left[\frac{e^{-t}}{-1} \right]_{\beta(10)^{\alpha}}^{\infty}$$

$$= -[0 - e^{-\beta(10)^{\alpha}}]$$

$$P(X > 10) = e^{-\beta(10)^{\alpha}}$$

2. The number of accidents in a year involving taxi drivers in a city follows a poisson distribution with mean equal to 3. Out of 1000 taxi drivers find approximately the number of drivers with (i) no accidents in a year (ii) more than 3 accidents in a year [$e^{-3} = 0.0498$]

Solution:

Average number of accidents in a year = 3 i. e., $\lambda = 3$

$n = 1000$

Let X denote the number of accidents in a year

$$\begin{aligned} \text{(i) } P(\text{no accident in year}) &= P(X=0) \\ &= e^{-3} \frac{3^0}{0!} \\ &= 0.0498 \end{aligned}$$

Out of 1000 drivers number drivers involved with no accidents

$$\begin{aligned} &= n \times P(X=0) \\ &= 1000 \times 0.0498 \\ &= 49.8 \\ &\sim 50 \end{aligned}$$

$$\begin{aligned} \text{(ii) } P(\text{more than 3 accidents}) &= P(X > 3) \\ &= 1 - P(X \leq 3) \\ &= 1 - \sum_0^3 P(X = x) \end{aligned}$$

$$\begin{aligned}
&= 1 - \sum_0^3 e^{-3} \frac{3^x}{x!} \\
&= 1 - e^{-3} \left[\frac{3^0}{0!} + \frac{3^1}{1!} + \frac{3^2}{2!} + \frac{3^3}{3!} \right] \\
&= 1 - e^{-3}(13) \\
&= 1 - 0.0498 \times 13 \\
&= 1 - 0.6474
\end{aligned}$$

$$P(X > 3) = 0.3526$$

Out of 1000 drivers, number of drivers involved with more than 3 accidents

$$\begin{aligned}
&= n \times P(X > 3) \\
&= 1000 \times 0.3526 \\
&= 352.6 \\
&\approx 353
\end{aligned}$$

3. If the number of incoming buses per minute at a bus terminus is a random variable having a poisson distribution with $\lambda = 0.9$, find the probability that there will be

- (i) Exactly 9 incoming buses during a period of 5 minutes
- (ii) Fewer than 10 incoming buses during a period of 8 minutes.
- (iii) At least 14 incoming buses during a period of 11 minutes.

Solution:

- (i) λ for number of incoming buses per minute = 0.9
 λ for number of incoming buses per 5 minutes = 0.9×5

$$= 4.5$$

$$P \text{ exactly 9 incoming buses during 5 minutes} = \frac{e^{-\lambda} \lambda^9}{9!}$$

$$P(X=9) = \frac{e^{-4.5} \times (4.5)^9}{9!}$$

(ii) Fewer than 10 incoming buses during a period of 8 minutes = $P(X < 10)$

$$\lambda = 0.9 \times 8 = 7.2$$

$$\text{Required probability} = \sum_{x=0}^9 \frac{e^{-7.2} \times (7.2)^x}{x!}$$

(iii) P atleast 14 incoming buses during a period of 11 minutes = $P(X \geq 14)$

$$= 1 - P(X < 14)$$

$$\lambda = 11 \times 0.9 = 9.9$$

$$\text{Required probability} = 1 - \sum_{x=0}^{13} \frac{e^{-9.9} \times (9.9)^x}{x!}$$

4. If X is normally distributed with mean 6 and standard deviation 5 find. (i) $P(0 \leq X \leq 8)$ (ii) $P(|X-6| < 10)$

Solution:

$$\text{Given } \mu = 6, \sigma = 5$$

$$Z = \frac{X - \mu}{\sigma}$$

$$X=0, Z = \frac{0-6}{5} = \frac{-6}{5} = -1.2$$

$$X=8, Z = \frac{8-6}{5} = \frac{2}{5} = 0.4$$

$$P(0 \leq X \leq 8) = P(-1.2 < Z < 0.4)$$

$$= P(0 < Z < 1.2) + P(0 < Z < 0.4)$$

$$= 0.3849 + 0.1554$$

$$= 0.5403$$

$$(ii) P(|X-6| < 10) = P(-10 < (X - 6) < 10)$$

$$X = -4, Z = \frac{-4-6}{5} = \frac{-10}{5} = -2$$

$$X = 16, Z = \frac{16-6}{5} = \frac{10}{5} = 2$$

$$P(-4 < X < 16) = P(-2 < Z < 2)$$

$$= 2P(0 < Z < 2)$$

$$= 2(0.4772)$$

$$= 0.9544$$

5. The mean score of 1000 students for examination is 34 and S.D is 16. (i) How many candidates can be expected to obtain marks between 30 and 60 assuming the normality of the distribution and (ii) determine the limit of the marks of the central 70% of the candidates.

Solution:

$$\mu = 34, \quad \sigma = 16, \quad N = 1000$$

$$(i) \quad P(30 < X < 60); \quad Z = \frac{X - \mu}{\sigma}$$

$$X = 30, Z_1 = \frac{30 - \mu}{\sigma} = \frac{30 - 34}{16} = \frac{-4}{16} = -0.25$$

$$Z_1 = -0.25$$

$$Z_2 = \frac{60-34}{16} = \frac{26}{16} = 1.625$$

$$Z_2 \approx 1.63$$

$$P(-0.25 < Z < 1.63) = P(0 < Z < 0.25) + P(0 < Z < 1.63)$$

$$= 0.0987 + 0.4484$$

$$= 0.5471$$

No of students scoring between 30 and 60 = 0.5471×1000

$$= 547$$

(ii) Limit of central 70% of candidates:

Value of Z_1 from the area table for the area 0.35 = -1.04

[as Z_1 lies to the left of $Z=0$]

Similarly $Z_2 = 1.04$

$$Z_1 = \frac{X-34}{16} = 1.04$$

$$X_1 = 16 \times 1.04 + 34$$

$$= 16.64 + 34$$

$$= 50.64$$

$$X_1 = 50.64$$

$$Z_2 = \frac{X-34}{16} = -1.04$$

$$X_2 = -1.04 \times 16 + 34$$

$$= -16.64 + 34$$

$$= 17.36$$

$$X_2 = 17.36$$

70% of the candidate score between 17.36 and 50.64

6. The air pressure in a randomly selected tyre put on a certain model new car is normally distributed with mean value 31 psi and standard deviation 0.2 psi.

- (i) What is the probability that the pressure for a randomly selected tyre (a) between 30.5 and 31.5 psi (b) between 30 and 32 psi
- (ii) What is the probability that the pressure for a randomly selected tyre exceeds 30.5 psi?

Solution:

Given $\mu = 31, \sigma = 0.2$

(i) (a) $P(30.5 < X < 31.5)$; $Z = \frac{X - \mu}{\sigma}$

(ii)

When $X = 30.5$, $Z = \frac{30.5 - 31}{0.2} = \frac{-0.5}{0.2} = -2.5$

When $X = 31.5$, $Z = \frac{31.5 - 31}{0.2} = \frac{0.5}{0.2} = 2.5$

$$\begin{aligned} P(30.5 < X < 31.5) &= P(-2.5 < Z < 2.5) \\ &= 2 P(0 < Z < 2.5) \\ &= 2(0.4938) \\ &= 0.9876 \end{aligned}$$

(b) $P(30 < X < 32)$

When $X = 30$, $Z = \frac{30 - 31}{0.2} = \frac{-1}{0.2} = -5$

$$\text{When } X=32, Z = \frac{32-31}{0.2} = \frac{1}{0.2} = 5$$

$$P(30 < X < 32) = P(-5 < Z < 5) = \text{area under the whole curve} = 1 \text{ (app.)}$$

$$\text{When } X=30.5, Z = \frac{30.5-31}{0.2} = \frac{-0.5}{0.2} = -2.5$$

$$\begin{aligned} P(X > 30.5) &= P(Z > -2.5) \\ &= 0.5 + P(0 < Z < 2.5) \\ &= 0.5 + 0.4938 \\ &= 0.9938 \end{aligned}$$

7. The mean weight of 500 male students in a certain college is 151 pounds and the standard deviation is 15 pounds. Assuming the weights are normally distributed, find how many students weigh (i) between 120 and 155 pounds (ii) more than 185 pounds.

Solution:

Let X denote the weight of male students.

$$\mu = 151, \quad \sigma = 15, \quad N = 500$$

$$(i) \quad \text{When } X=120, Z = \frac{120-151}{15} = \frac{-31}{15} = -2.067$$

$$\text{When } X=155, Z = \frac{155-151}{15} = \frac{4}{15} = 0.2667$$

$$\begin{aligned} P(120 < X < 155) &= P(-2.067 < Z < 0.2667) \\ &= P(-2.067 < Z < 0) + P(0 < Z < 0.2667) \\ &= P(0 < Z < 2.067) + P(0 < Z < 0.2667) \\ &= 0.4803 + 0.1026 \end{aligned}$$

$$P(120 < X < 155) = 0.5829$$

Probability for a students weigh lies between 120 and 155 is 0.5829

Out of 500 students number of students weigh lies

between 120 and 155 = 500×0.5829

=291 students

(ii) $P(X>185)$

$$\text{When } X= 185, Z= \frac{185 - 151}{15} = \frac{34}{15} = 2.2667$$

$$P(X>185) = P(Z > 2.2667)$$

$$= P(2.2667 < Z < \infty)$$

$$= P(0 < Z < \infty) - P(0 < Z < 2.2667)$$

$$= 0.5 - 0.4881$$

$$P(X>185) = 0.0119$$

Probability for a students weigh is above 185 is 0.0119

Out of 500 students number of students weigh

more than 185 pound = 500×0.0119

= 6 students

8. Find c , μ and σ^2 of the normal distribution whose probability function is given by $f(x) = c e^{x^2+3x}, -\infty < X < \infty$.

Solution:

$$f(x) = c e^{x^2+3x}, -\infty < X < \infty.$$

$$f(x) = c e^{-(x^2-3x)}$$

$$= c e^{-\left\{(x-\frac{3}{2})^2 - (\frac{3}{2})^2\right\}}$$

$$= c e^{-(x-\frac{3}{2})^2 + \frac{9}{4}}$$

$$= c e^{\frac{9}{4}} \cdot e^{-\frac{1}{2}\left(\frac{x-\frac{3}{2}}{1/\sqrt{2}}\right)^2}$$

$$\text{Mean} = \mu = \frac{3}{2} \text{ and } \sigma = \frac{1}{\sqrt{2}}, \quad \sigma^2 = \frac{1}{2}$$

The probability density function of a normal distribution is

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

$$\frac{1}{\sigma\sqrt{2\pi}} = c e^{\frac{9}{4}}$$

$$\Rightarrow c e^{\frac{9}{4}} = \frac{\sqrt{2}}{\sqrt{2\pi}}$$

$$\Rightarrow c = \frac{e^{-\frac{9}{4}}}{\sqrt{\pi}}$$

8. Obtain k , μ and σ^2 of the normal distribution whose probability distribution function is given by

$$f(x) = k e^{-2x^2+4x} \quad -\infty < X < \infty$$

Solution:

$$f(x) = k e^{-2x^2+4x} \quad -\infty < X < \infty$$

$$= k e^{-2(x^2-2x)}$$

$$= k e^{-2(x-1)^2+2}$$

$$= k e^2 e^{-2(x-1)^2}$$

$$= k e^2 e^{\frac{1}{2}(\frac{x-1}{1\sqrt{2}})^2}$$

$$\text{Mean} = \mu = 1 \text{ and } \sigma = \frac{1}{2}, \quad \sigma^2 = \frac{1}{4}$$

The probability density function of a normal distribution is

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

$$\frac{1}{\sigma\sqrt{2\pi}} = k e^2$$

$$\Rightarrow k e^2 = \frac{2}{\sqrt{2\pi}}$$

$$\Rightarrow k = \frac{\sqrt{2}e^{-2}}{\sqrt{\pi}}$$

10. A random variable X has the following probability mass function

X	0	1	2	3	4	5	6
$P(X=x)$	k	$3k$	$5k$	$7k$	$9k$	$11k$	$13k$

(1) Find k .

(2) Evaluate $P(X < 4)$, $P(X \geq 5)$ and $P(3 < X \leq 6)$

(3) What is the smallest value of x for which $P(X \leq x) > \frac{1}{2}$

Solution :

(1) Since $P(X = x)$ is a probability mass function

$$(2) \sum_{x=0}^6 P(X = x) = 1$$

(3)

$$P(X=0) + P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4) + P(X = 5) + P(X = 6) = 1.$$

$$\Rightarrow k + 3k + 5k + 7k + 9k + 11k + 13k = 1 \Rightarrow 49k = 1 \Rightarrow k = \frac{1}{49}$$

$$(2) P(X < 4) = P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3)$$

$$= \frac{1}{49} + \frac{3}{49} + \frac{5}{49} + \frac{7}{49} = \frac{16}{49}$$

$$P(X \geq 5) = P(X = 5) + P(X = 6) = \frac{11}{49} + \frac{13}{49} = \frac{24}{49}$$

$$P(3 < X \leq 6) = P(X = 4) + P(X = 5) + P(X = 6) = \frac{9}{49} + \frac{11}{49} + \frac{13}{49} = \frac{33}{49}$$

(3) The minimum value of x may be determined by trial and error method.

$$P(X \leq 0) = \frac{1}{49} < \frac{1}{2}$$

$$P(X \leq 1) = \frac{4}{49} < \frac{1}{2}$$

$$P(X \leq 2) = \frac{9}{49} < \frac{1}{2}$$

$$P(X \leq 3) = \frac{16}{49} < \frac{1}{2}$$

$$P(X \leq 4) = \frac{25}{49} > \frac{1}{2}$$

\therefore The smallest value of x for which $P(X \leq x) > \frac{1}{2}$ is 4

9. An urn contains 4 white and 3 red balls. Find the probability distribution of number of red balls in three draws one by one from the urn. (i) with replacement
(ii) without replacement

Solution : (i) with replacement

Let X be the random variable of drawing number of red balls in three draws.

$\therefore X$ can take the values 0,1,2,3.

$$P(\text{Red ball}) = \frac{3}{7} = P(R)$$

$$P(\text{Not Red ball}) = \frac{4}{7} = P(W)$$

$$\text{Therefore } P(X = 0) = P(www) = \frac{4}{7} \times \frac{4}{7} \times \frac{4}{7} = \frac{64}{343}$$

$$\begin{aligned} P(X = 1) &= P(Rww) + P(wRw) + P(wwR) \\ &= \frac{3}{7} \times \frac{4}{7} \times \frac{4}{7} + \frac{4}{7} \times \frac{3}{7} \times \frac{4}{7} + \frac{4}{7} \times \frac{4}{7} \times \frac{3}{7} \\ &= \frac{144}{343} \end{aligned}$$

$$\begin{aligned} P(X = 2) &= P(RRw) + P(RwR) + P(wRR) \\ &= \frac{3}{7} \times \frac{3}{7} \times \frac{4}{7} + \frac{3}{7} \times \frac{4}{7} \times \frac{3}{7} + \frac{4}{7} \times \frac{3}{7} \times \frac{3}{7} \\ &= \frac{108}{343} \end{aligned}$$

$$P(X=3) = P(RRR)$$

$$= \frac{3}{7} \times \frac{3}{7} \times \frac{3}{7}$$

$$= \frac{27}{343}$$

The required probability distribution is

X	0	1	2	3
P(X=x)	64/343	144/343	108/343	27/343

Clearly all p_i 's are ≥ 0 and $\sum p_i = 1$

Without replacement:

$$(i) \quad P(\text{no red ball}) = P(X=0) = \frac{{}^4C_3 \times {}^3C_0}{{}^7C_3} = \frac{4}{35}$$

$$(ii) \quad P(1 \text{ red ball}) = P(X=1) = \frac{{}^4C_2 \times {}^3C_1}{{}^7C_3} = \frac{18}{35}$$

$$(iii) \quad P(2 \text{ red ball}) = P(X=2) = \frac{{}^4C_1 \times {}^3C_2}{{}^7C_3} = \frac{12}{35}$$

$$(iv) \quad P(3 \text{ red ball}) = P(X=3) = \frac{{}^4C_0 \times {}^3C_3}{{}^7C_3} = \frac{1}{35}$$

X	0	1	2	3
P(X=x)	4/35	18/35	12/35	1/35

Clearly all p_i 's are ≥ 0 and $\sum p_i = 1$

11. The total life time (in year) of 5 year old dog of a certain breed is a Random Variable whose distribution function is given by

$$f(x) = \begin{cases} 0, & \text{for } x \leq 5 \\ 1 - \frac{25}{x^2}, & \text{for } x > 5 \end{cases}$$

Find the probability that such a five year old dog will live

(i) beyond 10 years (ii) less than 8 years

(iii) anywhere between 12 to 15 years.

Solution :

(i) $P(\text{dog living beyond 10 years})$

$$\begin{aligned} P(X > 10) &= 1 - P(X \leq 10) \\ &= 1 - \left(1 - \frac{25}{x^2}\right) \text{ when } x = 10 \\ &= 1 - \left(1 - \frac{25}{100}\right) = \frac{1}{4} \end{aligned}$$

(ii) $P(\text{dog living less than 8 years})$

$$\begin{aligned} P(X < 8) &= F(8) \text{ [since } P(X < 8) = P(X \leq 8) \text{ for a continuous distribution]} \\ &= \left(1 - \frac{25}{8^2}\right) = \left(1 - \frac{25}{64}\right) \\ &= \frac{39}{64} \end{aligned}$$

$$\begin{aligned}
 \text{(iii)} P(\text{dog living any where between 12 and 15 years}) &= P(12 < x < 15) \\
 &= F(15) - F(12) \\
 &= \left(1 - \frac{25}{8^2}\right) - \left(1 - \frac{25}{12^2}\right) \\
 &= \frac{1}{16}
 \end{aligned}$$

Six marks questions:

1. Find the probability distribution of the number of sixes in throwing three dice once.

Solution:

Let X denote the number of sixes in throwing three dice once. Then X is a random variable that can take the values 0, 1, 2 and 3

Probability of getting a six in a single die is $\frac{1}{6}$ and not getting a six is $\frac{5}{6}$

$$P(S) = \frac{1}{6}, P(\bar{S}) = \frac{5}{6}$$

$$\begin{aligned}
 P(X=0) &= P(\bar{S}\bar{S}\bar{S}) \\
 &= P(\bar{S})P(\bar{S})P(\bar{S}) \\
 &= \frac{5}{6} \frac{5}{6} \frac{5}{6} \\
 &= \frac{125}{216}
 \end{aligned}$$

$$P(X=1) = P(S\bar{S}\bar{S} \text{ or } \bar{S}S\bar{S} \text{ or } \bar{S}\bar{S}S)$$

$$\begin{aligned}
&= P(S \bar{S} \bar{S}) + P(\bar{S} S \bar{S}) + P(\bar{S} \bar{S} S) \\
&= P(S) P(\bar{S}) P(\bar{S}) + P(\bar{S}) P(S) P(\bar{S}) + P(\bar{S}) P(\bar{S}) P(S) \\
&= 3 \times \frac{1}{6} \frac{5}{6} \frac{5}{6} \\
&= \frac{75}{216}
\end{aligned}$$

$$\begin{aligned}
P(X=2) &= P(S S \bar{S} \text{ or } S \bar{S} S \text{ or } \bar{S} S S) \\
&= P(S S \bar{S}) + P(S \bar{S} S) + P(\bar{S} S S) \\
&= P(S) P(S) P(\bar{S}) + P(S) P(\bar{S}) P(S) + P(\bar{S}) P(S) P(S) \\
&= 3 \times \frac{1}{6} \frac{1}{6} \frac{5}{6} \\
&= \frac{15}{216}
\end{aligned}$$

$$\begin{aligned}
P(X=3) &= P(SSS) \\
&= P(S) P(S) P(S) \\
&= \frac{1}{6} \frac{1}{6} \frac{1}{6} \\
&= \frac{1}{216}
\end{aligned}$$

The required probability distribution is

X	0	1	2	3
P(X=x)	125/216	75/216	15/216	1/216

2. Two cards are drawn successively without replacement from a well shuffled pack of 52 cards. Find the probability distribution of the number of queens.

Solution:

Let X denote the number of queens obtained in a draw of two cards. Obviously X is a random variable which can take the values 0,1 and 2.

Queen	Others
-------	--------

4	48
---	----

$P(X=0) = P(\text{No queen})$

$$\begin{aligned} &= \frac{{}^4C_0 \times {}^{48}C_2}{{}^{52}C_2} \\ &= \frac{188}{221} \end{aligned}$$

$P(X=1) = P(\text{one queen})$

$$\begin{aligned} &= \frac{{}^4C_1 \times {}^{48}C_1}{{}^{52}C_2} \\ &= \frac{32}{221} \end{aligned}$$

$P(X=2) = P(\text{two queen})$

$$= \frac{{}^4C_2 \times {}^{48}C_0}{{}^{52}C_2} = \frac{1}{221}$$

The probability distribution function is

X 0 1 2

P(X=x) 188/221 32/221 1/221

3. Two bad oranges are accidentally mixed with ten good ones. Three oranges are drawn at random without replacement from this lot. Obtain the probability distribution for the number of bad oranges.

Solution:

Let X denote the number of bad oranges are drawn. Clearly X be a random variable which can take the values 0, 1 and 2. In this problem 'three oranges are drawn at random without replacement'

No good oranges	Bad oranges	Total
10	2	12

P(X=0) = (No bad oranges)

$$\begin{aligned} &= \frac{{}^2C_0 \times {}^{10}C_3}{{}^{12}C_3} \\ &= \frac{12}{22} \end{aligned}$$

P(X=1) = (one bad oranges)

$$\begin{aligned} &= \frac{{}^2C_1 \times {}^{10}C_2}{{}^{12}C_3} \\ &= \frac{9}{22} \end{aligned}$$

$$P(X=2) = (\text{two bad oranges})$$

$$= \frac{{}^2C_2 \times {}^{10}C_1}{{}^{12}C_3}$$

$$= \frac{1}{22}$$

The probability distribution is

x	0	1	2
P(X=x)	12/22	9/22	1/22

4. A discrete random variable X has the following probability distributions.

X	0	1	2	3	4	5	6	7	8
P(X=x)	a	3a	5a	7a	8a	11a	13a	15a	17a

(i) Find the value of a (ii) Find $P(x < 3)$ (iii) Find $P(3 < x < 7)$

Solution:

(i) Since $P(X=x)$ is a probability mass function

$$\sum_{x=0}^8 P(X = x) = 1$$

$$P(X=0) + P(X=1) + \dots + P(X=8) = 1$$

$$a + 3a + 5a + 7a + 8a + \dots + 17a = 1$$

$$81a = 1$$

$$a = \frac{1}{81}$$

$$\begin{aligned}
 \text{(ii)} \quad P(X < 3) &= P(X=0) + P(X=1) + P(X=2) \\
 &= a + 3a + 5a \\
 &= 9a \\
 &= \frac{9}{81} \\
 &= \frac{1}{9}
 \end{aligned}$$

$$\begin{aligned}
 \text{(iii)} \quad P(3 < X < 7) &= P(X=4) + P(X=5) + P(X=6) \\
 &= 9 + 11a + 13a \\
 &= 33a \\
 &= \frac{33}{81} \\
 &= \frac{11}{27}
 \end{aligned}$$

5. Find the p.d.f. $f(x) = \begin{cases} cx(1-x)^3 & 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$

Find (i) the constant c (ii) $P\left(x < \frac{1}{2}\right)$

Solution:

(i) Since $f(x)$ is a probability density function $\int_{-\infty}^{\infty} f(x) dx = 1$

$$c \int_0^1 x(1-x)^3 dx = 1$$

$$c \int_0^1 (1-x)x^3 dx = 1 \therefore \int_0^a f(x) dx = \int_0^a f(a-x) dx$$

$$c \int_0^1 (x^3 - x^4) dx = 1$$

$$c \left[\frac{x^4}{4} - \frac{x^5}{5} \right]_0^1 = 1$$

$$c \left[\frac{1}{4} - \frac{1}{5} \right] = 1$$

$$c \left(\frac{1}{20} \right) = 1$$

$$c = 20$$

$$\begin{aligned}
 \text{(ii)} \quad P\left(x < \frac{1}{2}\right) &= \int_{-\infty}^{1/2} f(x)dx = 20 \int_0^{1/2} x(1-x)^3 dx \\
 &= 20 \int_0^{1/2} (x - 3x^2 + 3x^3 - x^4)dx \\
 &= 20 \left[\frac{x^2}{2} - 3 \frac{x^3}{3} + 3 \frac{x^4}{4} - \frac{x^5}{5} \right]_0^{1/2} \\
 &= 20 \left[\frac{1}{8} - \frac{1}{8} + \frac{3}{64} - \frac{1}{160} \right] \\
 P\left(x < \frac{1}{2}\right) &= \frac{13}{16}
 \end{aligned}$$

6. For the distribution function given by

$$F(x) = \begin{cases} 0 & x < 0 \\ x^2 & 0 \leq x \leq 1 \\ 1 & x > 1 \end{cases}$$

Find the density function. Also evaluate (i) $P(0.5 < X < 0.75)$

(ii) $P(X \leq 0.5)$ (iii) $P(X > 0.75)$

Solution

$$f(x) = \frac{dF(x)}{dx} = \begin{cases} 2x, & 0 \leq x \leq 1 \\ 0 & \text{else where} \end{cases}$$

$$\begin{aligned}
 \text{(i)} \quad P(0.5 < x < 0.75) &= F(0.75) - F(0.5) \\
 &= (0.75)^2 - (0.5)^2 \\
 &= 0.3125
 \end{aligned}$$

$$\begin{aligned}
 \text{(ii)} \quad P(X \leq 0.5) &= P(-\infty < x \leq 0.5) \\
 &= F(0.5) - F(-\infty) \\
 &= (0.5)^2 - 0 \\
 &= 0.25
 \end{aligned}$$

$$\begin{aligned}
 \text{(iii)} \quad P(X > 0.75) &= P(0.75 \leq X < \infty) \\
 &= F(\infty) - F(0.75) \\
 &= 1 - (0.75)^2 \\
 &= 0.4375
 \end{aligned}$$

7. A random variable X has a probability density function

$$f(x) = \begin{cases} k, & 0 < x < 2\pi \\ 0 & \text{else where} \end{cases}$$

Find (i) k (ii) $P\left(0 < X < \frac{\pi}{2}\right)$ (iii) $P\left(\frac{\pi}{2} < X < \frac{3\pi}{2}\right)$

(i) Since f(x) is a pdf $\int_{-\infty}^{\infty} f(x) dx = 1$

$$k \int_0^{2\pi} dx = 1$$

$$k[x]_0^{2\pi} = 1$$

$$2\pi k = 1$$

$$\Rightarrow k = \frac{1}{2\pi}$$

$$\begin{aligned}
 \text{(ii)} \quad P\left(0 < X < \frac{\pi}{2}\right) &= \int_0^{\pi/2} f(x) dx = \int_0^{\pi/2} \frac{1}{2\pi} dx \\
 &= \frac{1}{2\pi} [x]_0^{\pi/2} \\
 &= \frac{1}{2\pi} \times \frac{\pi}{2} \\
 &= \frac{1}{4}
 \end{aligned}$$

$$\begin{aligned}
 \text{(iii)} \quad P\left(\frac{\pi}{2} < X < \frac{3\pi}{2}\right) &= \int_{\pi/2}^{3\pi/2} f(x) dx = \frac{1}{2\pi} \int_{\pi/2}^{3\pi/2} dx \\
 &= \frac{1}{2\pi} [x]_{\pi/2}^{3\pi/2} \\
 &= \frac{1}{2\pi} \left(3\frac{\pi}{2} - \frac{\pi}{2}\right) \\
 &= \frac{1}{2\pi} \times \pi \\
 &= \frac{1}{2}
 \end{aligned}$$

8. Find the probability mass function, and the cumulative distribution function for getting '3's when two dice are thrown.

Solution :

Two dice are thrown. Let X be the random variable of getting number of '3's. Therefore X can take the values 0, 1, 2.

$$P(\text{no '3'}) = P(X = 0) = \frac{25}{36}$$

$$P(\text{one '3'}) = P(X = 1) = \frac{10}{36}$$

$$P(\text{two '3's}) = P(X = 2) = \frac{1}{36}$$

Sample Space

(1,1) (1,2) **(1,3)** (1,4) (1,5) (1,6)

(2,1) (2,2) **(2,3)** (2,4) (2,5) (2,6)

(3,1) **(3,2)** **(3, 3)** **(3,4)** **(3,5)** **(3,6)**

(4,1) (4,2) **(4,3)** (4,4) (4,5) (4,6)

(5,1) (5,2) **(5,3)** (5,4) (5,5) (5,6)

(6,1) (6,2) **(6,3)** (6,4) (6,5) (6,6)

probability mass function is given by

x	0	1	2
$P(X = x)$	$25/36$	$10/36$	$1/36$

Cumulative distribution function :

We have $F(x) = \sum_{x_i}^x P(X = x_i)$

$$F(0) = P(X = 0) = \frac{25}{36}$$

$$F(1) = P(X = 0) + P(X = 1) = \frac{35}{36}$$

$$F(2) = P(X = 0) + P(X = 1) + P(X = 2) = \frac{36}{36} = 1$$

X	0	1	2
$F(x)$	$25/36$	$35/36$	1

9. A continuous random variable X follows the probability law,

$$f(x) = \begin{cases} kx(1-x)^{10}, & 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$$

Find k

Solution :

Since $f(x)$ is a p.d.f $\int_{-\infty}^{\infty} f(x) dx = 1$

$$\int_0^1 kx(1-x)^{10} dx = 1$$

$$\int_0^1 k(1-x)[1-(1-x)]^{10} dx = 1$$

$$k \int_0^1 (1-x)x^{10} dx = 1$$

$$k \int_0^1 (x^{10} - x^{11}) dx = 1$$

$$k \left[\frac{x^{11}}{11} - \frac{x^{12}}{12} \right]_0^1 = 1$$

$$k \left[\frac{1}{11} - \frac{1}{12} \right] = 1$$

$$k = 132$$

10. A continuous random variable X has p.d.f.

$f(x) = 3x^2$, $0 \leq x \leq 1$, Find a and b such that. (i) $P(X \leq a) = P(X > a)$ and

(ii) $P(X > b) = 0.05$

Solution :

(i) Since the total probability is 1, [Given that $P(X \leq a) = P(X > a)$]

$$P(X \leq a) + P(X > a) = 1$$

$$\text{i.e., } P(X \leq a) + P(X \leq a) = 1$$

$$\Rightarrow P(X \leq a) = \frac{1}{2}$$

$$\int_0^a f(x) dx = \frac{1}{2} \Rightarrow \int_0^a 3x^2 dx = \frac{1}{2}$$

$$\left[\frac{3x^3}{3} \right]_0^a = \frac{1}{2}$$

$$\Rightarrow a^3 = \frac{1}{2}$$

$$\text{i.e., } a = \left(\frac{1}{2} \right)^{1/3}$$

$$(ii) P(X > b) = 0.05$$

$$\therefore \int_b^1 f(x) dx = 0.05$$

$$\Rightarrow \int_b^1 3x^2 dx = 0.05$$

$$\left[\frac{3x^3}{3} \right]_b^1 = 0.05 \Rightarrow 1 - b^3 = 0.05$$

$$b^3 = 1 - 0.05 = 0.95$$

$$= \frac{95}{100}$$

$$\Rightarrow b = \left(\frac{19}{20} \right)^{1/3}$$

$$11. \text{If } f(x) = \begin{cases} \frac{A}{x}, & 1 < x < e^3 \\ 0, & \text{elsewhere} \end{cases}$$

is a probability density function of a continuous random variable X , find $p(x > e)$

Solution:

Since $f(x)$ is a p.d.f. $\int_{-\infty}^{\infty} f(x) dx = 1$

$$\int_1^{e^3} \frac{A}{x} dx = 1 \Rightarrow A [\log x]_1^{e^3} = 1$$

$$\Rightarrow A [\log e^3 - \log 1] = 1$$

$$\Rightarrow A [3] = 1$$

$$\Rightarrow A = 1/3$$

Therefore $f(x) = \begin{cases} \frac{A}{x}, & 1 < x < e^3 \\ 0, & \text{elsewhere} \end{cases}$

$$P(x > e) = \frac{1}{3} \int_e^{e^3} \frac{1}{x} dx$$

$$= \frac{1}{3} [\log x]_e^{e^3}$$

$$= \frac{1}{3} (3 - 1)$$

$$= \frac{2}{3}$$

12. Two unbiased dice are thrown together at random. Find the expected value of the total number of points shown up.

Solution :

Let X be the random variable which represents the sum of the numbers shown in the two dice. If both show one then the sum total is 2. If both show six then the sum is 12.

The random variable X can take values from 2 to 12.

(1, 1)

(1, 2) (2, 1)

(1, 3) (2, 2) (3, 1)

(1, 4) (2, 3) (3, 2) (4, 1)

(1, 5) (2, 4) (3, 3) (4, 2) (5, 1)

(1, 6) (2, 5) (3, 4) (4, 3) (5, 2) (6, 1)

(2, 6) (3, 5) (4, 4) (5, 3) (6, 2)

(3, 6) (4, 5) (5, 4) (6, 3)

(4, 6) (5, 5) (6, 4)

(5, 6) (6, 5)

(6, 6)

∴ The probability distribution is given by.

X	2	3	4	5	6	7	8	9	10	11	12
P(X=x)	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

$$\begin{aligned}
 E(X) &= \sum p_i x_i = \sum x_i p_i \\
 &= \left(2 \times \frac{1}{36}\right) + \left(3 \times \frac{2}{36}\right) + \left(4 \times \frac{3}{36}\right) + \dots + \left(12 \times \frac{1}{36}\right) \\
 &= \frac{252}{36} \\
 &= 7
 \end{aligned}$$

13. The probability of success of an event is p and that of failure is q . Find the expected number of trials to get a first success.

Solution:

Let X be the random variable denoting ‘Number of trials to get a first success’. The success can occur in the 1st trial. ∴ The probability of success in the 1st trial is p . The success in the 2nd trial means failure in the 1st trial.

∴ Probability is qp .

Success in the 3rd trial means failure in the first two trials. ∴ Probability of success in the 3rd trial is q^2p . As it goes on, the success may occur in the n th trial which mean the first $(n - 1)$ trials are failures.

∴ probability = $q^{n-1}p$.

The probability distribution is as follows

X	1	2	3	$n \dots$
$P(x)$	p	qp	q^2p	$q^{n-1}p \dots$

$$\begin{aligned} \therefore E(X) &= \sum p_i x_i \\ &= 1 \cdot p + 2qp + 3q^2p + \dots + nq^{n-1}p \dots \\ &= p[1 + 2q + 3q^2 + \dots + nq^{n-1} + \dots] \\ &= p[1 - q]^{-2} \\ &= p(p)^{-2} \\ &= \frac{p}{p^2} \\ &= \frac{1}{p} \end{aligned}$$

14. An urn contains 4 white and 3 Red balls. Find the probability distribution of the number of red balls in three draws when a ball is drawn at random with replacement. Also find its mean and variance.

Solution :

with replacement

Let X be the random variable of drawing number of red balls in three draws.

$\therefore X$ can take the values 0,1,2,3.

$$P(\text{Red ball}) = \frac{3}{7} = P(R)$$

$$P(\text{Not Red ball}) = \frac{4}{7} = P(W)$$

$$\begin{aligned}\text{Therefore } P(X = 0) &= P(www) = \frac{4}{7} \times \frac{4}{7} \times \frac{4}{7} \\ &= \frac{64}{343}\end{aligned}$$

$$\begin{aligned}P(X = 1) &= P(Rww) + P(wRw) + P(wwR) \\ &= \frac{3}{7} \times \frac{4}{7} \times \frac{4}{7} + \frac{4}{7} \times \frac{3}{7} \times \frac{4}{7} + \frac{4}{7} \times \frac{4}{7} \times \frac{3}{7} \\ &= \frac{144}{343}\end{aligned}$$

$$\begin{aligned}P(X = 2) &= P(RRw) + P(RwR) + P(wRR) \\ &= \frac{3}{7} \times \frac{3}{7} \times \frac{4}{7} + \frac{3}{7} \times \frac{4}{7} \times \frac{3}{7} + \frac{4}{7} \times \frac{3}{7} \times \frac{3}{7} \\ &= \frac{108}{343}\end{aligned}$$

$$\begin{aligned}P(X=3) &= P(RRR) \\ &= \frac{3}{7} \times \frac{3}{7} \times \frac{3}{7} \\ &= \frac{27}{343}\end{aligned}$$

The required probability distribution is

X	0	1	2	3
P(X=x)	64/343	144/343	108/343	27/343

Clearly all p_i 's are ≥ 0 and $\sum p_i = 1$

$$\text{Mean } E(X) = \sum p_i x_i$$

$$\begin{aligned} &= 0\left(\frac{64}{343}\right) + 1\left(\frac{144}{343}\right) + 2\left(\frac{108}{343}\right) + 3\left(\frac{27}{343}\right) \\ &= \frac{9}{7} \end{aligned}$$

$$\text{Variance} = E(X^2) - [E(X)]^2$$

$$E(X^2) = \sum p_i x_i^2$$

$$\begin{aligned} &= 0^2 \left(\frac{64}{343}\right) + 1^2 \left(\frac{144}{343}\right) + 2^2 \left(\frac{108}{343}\right) + 3^2 \left(\frac{27}{343}\right) \\ &= \frac{117}{49} \end{aligned}$$

$$\begin{aligned} \text{Variance} &= \frac{117}{49} - \left(\frac{9}{7}\right)^2 \\ &= \frac{36}{49} \end{aligned}$$

15. A game is played with a single fair die, A player wins Rs. 20 if a 2 turns up, Rs. 40 if a 4 turns up, loses Rs. 30 if a 6 turns up. While he neither wins nor loses if any other face turns up. Find the expected sum of money he can win.

Solution :

Let X be the random variable denoting the amount he can win. The possible values of X are 20, 40, -30 and 0.

$$P[X = 20] = P(\text{getting } 2) = \frac{1}{6}$$

$$P[X = 40] = P(\text{getting 4}) = \frac{1}{6}$$

$$P[X = -30] = P(\text{getting 6}) = \frac{1}{6}$$

The remaining probability is $\frac{1}{2}$

X	20	40	-30	0
$P(x)$	1/6	1/6	1/6	1/2

$$\text{Mean } E(X) = \sum p_i x_i$$

$$= 20 \left(\frac{1}{6}\right) + 40 \left(\frac{1}{6}\right) + (-30) \left(\frac{1}{6}\right) + 0 \left(\frac{1}{2}\right)$$

$$= 5$$

Expected sum of money he can win = Rs. 5

16. In a continuous distribution the p.d.f of X is

$$f(x) = \begin{cases} \frac{3}{4} x(2-x), & 0 < x < 2 \\ 0 & \text{elsewhere} \end{cases}$$

Find the mean and the variance of the distribution.

Solution :

$$\text{Mean} = E(X) = \int_{-\infty}^{\infty} x f(x) dx$$

$$= \int_0^2 x \frac{3}{4} x(2-x) dx$$

$$= \frac{3}{4} \int_0^2 x^2 (2 - x) dx$$

$$= \frac{3}{4} \int_0^2 (2x^2 - x^3) dx$$

$$= \frac{3}{4} \left[2 \frac{x^3}{3} - \frac{x^4}{4} \right]_0^2$$

$$= \frac{3}{4} \left[\frac{2}{3} (8) - \frac{16}{4} \right]$$

$$= 1$$

Mean = 1

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x) dx$$

$$= \int_0^2 x^2 \frac{3}{4} x(2 - x) dx$$

$$= \frac{3}{4} \int_0^2 x^3 (2 - x) dx$$

$$= \frac{3}{4} \int_0^2 (2x^3 - x^4) dx$$

$$= \frac{3}{4} \left[2 \frac{x^4}{4} - \frac{x^5}{5} \right]_0^2$$

$$= \frac{3}{4} \left[\frac{16}{2} - \frac{32}{5} \right]$$

$$E(X^2) = \frac{6}{5}$$

$$\text{Variance} = E(X^2) - [E(X)]^2$$

$$= \frac{6}{5} - 1$$

$$= \frac{1}{5}$$

17. Find the mean and variance of the distribution

$$f(x) = \begin{cases} 3e^{-3x}, & 0 < x < \infty \\ 0, & \text{elsewhere} \end{cases}$$

Solution:

$$\text{Mean} = E(X) = \int_{-\infty}^{\infty} x f(x) dx = \int_0^{\infty} x (3 e^{-3x}) dx$$

$$= 3 \int_0^{\infty} x e^{-3x} dx = 3 \cdot \frac{1!}{3^2} = \frac{1}{3} \quad (\text{by Gamma integral})$$

$$\text{Mean} = \frac{1}{3}$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x) dx = \int_0^{\infty} x^2 (3 e^{-3x}) dx$$

$$= 3 \int_0^{\infty} x^2 e^{-3x} dx$$

$$= 3 \cdot \frac{2!}{3^3} \quad (\text{by Gamma integral})$$

$$= \frac{2}{9}$$

$$V(X) = E(X^2) - [E(X)]^2$$

$$= \frac{2}{9} - \left(\frac{1}{3}\right)^2 = \frac{1}{9}$$

$$\text{Mean} = \frac{1}{3} \quad ; \quad \text{Variance} = \frac{1}{9}$$

18. A die is tossed twice. A success is getting an odd number on a toss. Find the mean and the variance of the probability distribution of the number of successes.

Solution:

Let X denote the number of success in throwing a die twice.

X is clearly a random variable and takes the values 0, 1 and 2

$$P(\text{success}) = \frac{3}{6} ; P(\text{failure}) = \frac{3}{6}$$

$$P(S) = \frac{1}{2} , P(F) = \frac{1}{2}$$

$$P(X=0) = P(FF) = P(F) P(F) = \frac{1}{2} \frac{1}{2} = \frac{1}{4}$$

$$\begin{aligned} P(X=1) &= P(SF \text{ or } FS) = P(SF) + P(FS) = P(S) P(F) + P(F) P(S) \\ &= 2 P(F) P(S) \\ &= 2 \cdot \frac{1}{2} \frac{1}{2} \\ &= \frac{1}{2} \end{aligned}$$

$$P(X=2) = P(SS) = P(S) P(S) = \frac{1}{2} \frac{1}{2} = \frac{1}{4}$$

Probability mass function is

X	0	1	2
P(X=x)	1/4	1/2	1/4

$$\text{Mean} = E[X] = \sum_{-\infty}^{\infty} x_i p_i = 0\left(\frac{1}{4}\right) + 1 \cdot \frac{1}{2} + 2 \frac{1}{4} = 1$$

$$\text{Variance} = E(X^2) - [E(X)]^2$$

$$E(X^2) = \sum_{-\infty}^{\infty} x_i^2 p_i$$

$$= (0^2) \left(\frac{1}{4}\right) + (1^2) \left(\frac{1}{2}\right) + (2^2) \left(\frac{1}{4}\right)$$

$$= \frac{3}{2}$$

$$V(X) = E(X^2) - [E(X)]^2$$

$$= \frac{3}{2} - (1)^2$$

$$= \frac{1}{2}$$

$$\text{Mean} = 1, \text{ Variance} = \frac{1}{2}$$

19. Find the expected value of the number on a die when thrown.

Solution:

Let x denote the number on die. Clearly X can take the values 1, 2, 3, 4, 5 and 6

The probability mass function is

X	1	2	3	4	5	6
$P(X = x)$	1/6	1/6	1/6	1/6	1/6	1/6

$$\text{Mean} = E[X] = \sum_{-\infty}^{\infty} x_i p_i$$

$$= (1) \left(\frac{1}{6}\right) + (2) \left(\frac{1}{6}\right) + (3) \left(\frac{1}{6}\right) + (4) \left(\frac{1}{6}\right) + (5) \left(\frac{1}{6}\right) + (6) \left(\frac{1}{6}\right)$$

$$\begin{aligned} &= \frac{21}{6} \\ &= 3.5 \end{aligned}$$

20. In an entrance examination a student has to answer all the 120 questions. Each question has four options and only one option is correct. A student gets 1 mark for a correct answer and loses half mark for a wrong answer. What is the expectation of the mark scored by a student if he chooses the answer to each question at random?

Solution:

Let x be the mark obtained by a student for answering a question.

X is a random variable and can take the values $1, -\frac{1}{2}$

$$P(X=1) = P(\text{answering a question correctly}) = \frac{1}{4}$$

$$P(X=-\frac{1}{2}) = P(\text{answering question wrongly}) = \frac{3}{4}$$

Probability mass function is

$$X \quad 1 \quad -1/2$$

$$P(X=x) \quad \frac{1}{4} \quad \frac{3}{4}$$

$$\begin{aligned} E[X] &= \sum x_i p_i \\ &= (1) \cdot \left(\frac{1}{4}\right) + \left(-\frac{1}{2}\right) \left(\frac{3}{4}\right) \\ &= -\frac{1}{8} \end{aligned}$$

Expectation of mark when he answering single question is $-\frac{1}{8}$

$$\begin{aligned}\text{Expectation of marks for answering 120 questions} &= (120) \left(-\frac{1}{8} \right) \\ &= -15\end{aligned}$$

21. Two cards are drawn with replacement from a well shuffled deck of 52 cards. Find the mean and variance for the number of aces

Solution:

Let X denote the number of aces drawn in drawing two cards with replacement .

X is a random variable takes the values 0, 1 and 2

$$P(A) = \frac{4}{52} ; P(\bar{A}) = \frac{48}{52}$$

$$\begin{aligned}P(X=0) &= P(\text{no aces}) = P(\bar{A}\bar{A}) \\ &= P(\bar{A}) P(\bar{A}) \\ &= \frac{48}{52} \frac{48}{52} \\ &= \frac{144}{169}\end{aligned}$$

$$\begin{aligned}P(X=1) &= P(\text{one ace}) = P(A\bar{A} \text{ or } \bar{A}A) \\ &= P(A) P(\bar{A}) + P(\bar{A}) P(A) \\ &= 2 P(A) P(\bar{A}) \\ &= 2 \cdot \frac{4}{52} \frac{48}{52} \\ &= \frac{24}{169}\end{aligned}$$

$$\begin{aligned}
 P(X=2) &= P(\text{two aces}) = P(AA) \\
 &= P(A) P(A) \\
 &= \frac{4}{52} \frac{4}{52} \\
 &= \frac{1}{169}
 \end{aligned}$$

X	0	1	2
P(X=x)	144/169	24/169	1/169

$$\begin{aligned}
 \text{Mean} = E[X] &= \sum_{-\infty}^{\infty} x_i p_i \\
 &= (0) \left(\frac{144}{169}\right) + (1) \left(\frac{24}{169}\right) + (2) \left(\frac{1}{169}\right) \\
 &= \frac{2}{13}
 \end{aligned}$$

$$\begin{aligned}
 E(X^2) &= \sum_{-\infty}^{\infty} x_i^2 p_i \\
 &= (0^2) \left(\frac{144}{169}\right) + (1^2) \left(\frac{24}{169}\right) + (2^2) \left(\frac{1}{169}\right) \\
 &= \frac{28}{169}
 \end{aligned}$$

$$\begin{aligned}
 V(X) &= E(X^2) - [E(X)]^2 \\
 &= \frac{28}{169} - \left(\frac{2}{13}\right)^2 \\
 &= \frac{24}{169}
 \end{aligned}$$

22. In a gambling game a man wins Rs.10 if he gets all heads or all tails and loses Rs.5 if he gets 1 or 2 heads when 3 coins are tossed once. Find his expectation of gain.

Solution:

Let X denote amount

X is a random variable takes the values 10 and -5.

The sample space for 3 coins are tossed is

{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT}

$$P(X= 10) = P(\text{getting 3 heads or 3 tails}) = \frac{2}{8}$$

$$P(X=-5) = P(\text{ getting 1 head or 2 heads}) = \frac{6}{8}$$

Probability mass function is

X	10	-5
P(X=x)	2/8	6/8

$$\begin{aligned}\text{Expectation of gain} &= E[X] = \sum_{-\infty}^{\infty} x_i p_i \\ &= (10) \left(\frac{2}{8}\right) + (-5) \left(\frac{6}{8}\right) \\ &= -1.25\end{aligned}$$

Loss of Rs. 1.25

23. The probability distribution of a random variable X is given below:

X	0	1	2	3
P(X=x)	0.1	0.3	0.5	0.1

$X^2 + 2X$ find the mean and variance of Y

Solution:

When $Y = X^2 + 2X$, the distribution function is

$Y = X^2 + 2X$	0	3	8	15
$p_i = P(Y=y)$	0.1	0.3	0.5	0.1

$$\begin{aligned}\text{Mean} &= E[Y] = \sum_{-\infty}^{\infty} y_i p_i \\ &= (0)(0.1) + (3)(0.3) + (8)(0.5) + (15)(0.1) \\ &= 6.4\end{aligned}$$

$$E[Y^2] = (0^2)(0.1) + (3^2)(0.3) + (8^2)(0.5) + (15^2)(0.1) = 57.5$$

$$\begin{aligned}V(Y) &= E(Y^2) - [E(Y)]^2 \\ &= (57.5) - (6.4)^2 \\ &= 16.24\end{aligned}$$

24. Find the Mean and Variance for the following probability density functions

$$(i) \quad f(x) = \begin{cases} \frac{1}{24} & , -12 \leq x \leq 12 \\ 0 & , \textit{elsewhere} \end{cases}$$

Solution:

$$\text{Mean} = E(X) = \int_{-\infty}^{\infty} x f(x) dx$$

$$= \int_{-12}^{12} x \left(\frac{1}{24} \right) dx$$

$$\left(\int_{-a}^a f(x) dx = 0 \textit{ when } f(x) \textit{ is an odd function} \right)$$

$$= 0$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x) dx$$

$$= \int_{-12}^{12} x^2 \left(\frac{1}{24} \right) dx$$

$$= 2 \cdot \frac{1}{24} \int_0^{12} x^2 dx$$

$$= \frac{1}{12} \left[\frac{x^3}{3} \right]_0^{12}$$

$$= \frac{1}{12} \left[\frac{12^3}{3} \right]$$

$$= \frac{144}{3}$$

$$= 48$$

$$V(X) = E(X^2) - [E(X)]^2$$

$$= 48 - 0$$

$$= 48$$

$$(ii) \quad f(x) = \begin{cases} \alpha e^{-\alpha x}, & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

Solution:

$$\text{Mean} = E(X) = \int_{-\infty}^{\infty} x f(x) dx = \int_0^{\infty} x \alpha e^{-\alpha x} dx$$

$$= \alpha \int_0^{\infty} x e^{-\alpha x} dx = \alpha \cdot \frac{1!}{\alpha^2} = \frac{1}{\alpha} \quad (\text{by Gamma integral})$$

$$\text{Mean} = \frac{1}{\alpha}$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x) dx = \int_0^{\infty} x^2 \alpha e^{-\alpha x} dx$$

$$= \alpha \int_0^{\infty} x^2 e^{-\alpha x} dx = \alpha \cdot \frac{2!}{\alpha^3} = \frac{2}{\alpha^2} \quad (\text{by Gamma integral})$$

$$V(X) = E(X^2) - [E(X)]^2$$

$$= \frac{2}{\alpha^2} - \frac{1}{\alpha^2}$$

$$\text{Variance} = \frac{1}{\alpha^2}$$

$$(iii) f(x) = \begin{cases} xe^{-x}, & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

Solution:

$$\text{Mean} = E(X) = \int_{-\infty}^{\infty} x f(x) dx = \int_0^{\infty} x (x e^{-x}) dx$$

$$= \int_0^{\infty} x^2 e^{-x} dx$$

$$= \frac{2!}{1^3} \text{ (by Gamma integral)}$$

$$= 2$$

$$\text{Mean} = 2$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x) dx = \int_0^{\infty} x^2 (x e^{-x}) dx$$

$$= \int_0^{\infty} x^3 e^{-x} dx$$

$$= \frac{3!}{1^4} \text{ (by Gamma integral)}$$

$$= 6$$

$$\begin{aligned}V(X) &= E(X^2) - [E(X)]^2 \\ &= 6 - 2^2 \\ &= 6 - 4\end{aligned}$$

Mean = 2; Variance = 2

25. Let X be a binomially distributed variable with mean 2 and standard deviation $\frac{2}{\sqrt{3}}$. Find the corresponding probability function.

Solution :

$$np = 2 ; npq = \frac{2}{\sqrt{3}}$$

$$\therefore npq = 4/3$$

$$\therefore q = \frac{npq}{np} = \frac{4/3}{2} = \frac{4}{6} = \frac{2}{3}$$

$$\therefore p = 1 - q = 1 - \frac{2}{3} = \frac{1}{3}$$

$$np = 2 \therefore n\left(\frac{1}{3}\right) = 2 \Rightarrow n = 6$$

\therefore The probability function for the distribution is

$$P[X = x] = {}^6C_x \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{6-x}, \quad x = 0, 1, 2, \dots, 6$$

26. A pair of dice is thrown 10 times. If getting a doublet is considered a success find the probability of (i) 4 success (ii) No success.

Solution :

$n = 10$. A doublet can be obtained when a pair of dice thrown is $\{(1,1), (2,2) (3,3), (4,4), (5,5) (6,6)\}$ ie., 6 ways.

Probability of success is getting a doublet

$$\therefore p = \frac{6}{36} = \frac{1}{6}$$

$$q = 1 - p = 1 - \frac{1}{6} = \frac{5}{6}$$

Let X be the number of success.

We have $P[X = x] = nC_x p^x q^{n-x}$

$$\begin{aligned} \text{(a) } P(4 \text{ successes}) &= P[X = 4] = {}^{10}C_4 \left(\frac{1}{6}\right)^4 \left(\frac{5}{6}\right)^6 \\ &= \frac{210 \times 5^6}{6^{10}} \\ &= \frac{35}{216} \left(\frac{5}{6}\right)^6 \end{aligned}$$

$$\begin{aligned} \text{(b) } P(\text{no success}) &= P(X = 0) \\ &= {}^{10}C_0 \left(\frac{5}{6}\right)^{10} \\ &= \left(\frac{5}{6}\right)^{10} \end{aligned}$$

27. If a publisher of non-technical books takes a great pain to ensure that his books are free of typological errors, so that the probability of any given page containing atleast one such error is 0.005 and errors are independent from page to page (i) what is the probability that one of its 400 page novels will contain exactly one page with error. (ii) atleast three pages with errors. [$e^{-2} = 0.1353$; $e^{-0.2} = 0.819$].

Solution :

$$n = 400 , p = 0.005$$

$$\therefore np = 2 = \lambda$$

$$\begin{aligned} \text{(i) } P(\text{one page with error}) &= P(X = 1) = \frac{e^{-\lambda} \lambda^1}{1!} = \frac{e^{-2} 2^1}{1!} \\ &= 0.1363 \times 2 = 0.2726 \end{aligned}$$

$$\text{(ii) } P(\text{atmost 3 pages with error}) = P(X \leq 3)$$

$$\begin{aligned} &= \sum_{x=0}^3 \frac{e^{-\lambda} \lambda^x}{x!} \\ &= \sum_0^3 \frac{e^{-2} 2^x}{x!} \\ &= e^{-2} \left[1 + \frac{2}{1!} + \frac{2^2}{2!} + \frac{2^3}{3!} \right] \\ &= e^{-2} \left(\frac{19}{3} \right) \\ &= 0.8569 \end{aligned}$$

28. Suppose that the probability of suffering a side effect from a certain vaccine is 0.005. If 1000 persons are inoculated, find approximately the probability that (i) atmost 1 person suffer. (ii) 4, 5 or 6 persons suffer.

$$[e^{-5} = 0.0067]$$

Solution :

Let the probability of suffering from side effect be p

$$n = 1000, p = 0.005, \lambda = np = 5.$$

$$(i) P(\text{atmost 1 person suffer}) = p(X \leq 1)$$

$$= p(X = 0) + p(X = 1)$$

$$= \frac{e^{-\lambda} \lambda^0}{0!} + \frac{e^{-\lambda} \lambda^1}{1!}$$

$$= e^{-\lambda} [1 + \lambda]$$

$$= e^{-5} (1 + 5)$$

$$= 6 \times e^{-5}$$

$$= 6 \times 0.0067$$

$$= 0.0402$$

$$(ii) P(4, 5 \text{ or } 6 \text{ persons suffer}) = p(X = 4) + p(X = 5) + p(X = 6)$$

$$= \frac{e^{-\lambda} \lambda^4}{4!} + \frac{e^{-\lambda} \lambda^5}{5!} + \frac{e^{-\lambda} \lambda^6}{6!}$$

$$= \frac{e^{-\lambda} \lambda^4}{4!} \left[1 + \frac{\lambda}{5} + \frac{\lambda^2}{30} \right]$$

$$= \frac{e^{-5} 5^4}{24} \left[1 + \frac{5}{5} + \frac{25}{30} \right]$$

$$= \frac{e^{-5} 5^4}{24} \left(\frac{17}{6} \right)$$

$$= \frac{10625}{144} \times 0.0067$$

$$= 0.4944$$

29. In a Poisson distribution if $P(X = 2) = P(X = 3)$ find $P(X = 5)$

[given $e^{-3} = 0.050$].

Solution :

Given $P(X = 2) = P(X = 3)$

$$\therefore \frac{e^{-\lambda} \lambda^2}{2!} = \frac{e^{-\lambda} \lambda^3}{3!}$$

$$\Rightarrow 3\lambda^2 = \lambda^3$$

$$\lambda^2 (3 - \lambda) = 0 \quad \text{As } \lambda \neq 0, \lambda = 3$$

$$P(X=5) = \frac{e^{-\lambda} \lambda^5}{5!}$$

$$= \frac{e^{-3} 3^5}{5!}$$

$$= \frac{0.050 \times 243}{120}$$

$$= 0.101$$

30. Four coins are tossed simultaneously. What is the probability of getting (a) exactly 2 heads (b) at least two heads (c) at most two heads.

Solution :

Given that $n=4$

$$P = P(\text{getting head when a coin when a coin is tossed}) = \frac{1}{2}, q = \frac{1}{2}$$

Let X denote the number of heads,

$\therefore X$ is a random variable follows binomial distribution

$$P(X=x) = nC_x p^x q^{n-x} \quad x=0, 1, 2, 3, 4$$

(a) Exactly 2 heads

$$P(x=2) = 4C_2 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^{4-2}$$

(b) At least 2 heads

$$\begin{aligned} P(X \geq 2) &= 1 - P(X < 2) \\ &= 1 - [P(X=0) + P(X=1)] \\ &= 1 - \left[4C_0 \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^4 + 4C_1 \left(\frac{1}{2}\right)^1 \left(\frac{1}{2}\right)^3 \right] \\ &= 1 - \left[\frac{1}{16} + \frac{4}{16} \right] \\ &= \frac{11}{16} \end{aligned}$$

(c) Atmost 2 heads

$$\begin{aligned}P(X \leq 2) &= 1 - P(X > 2) \\&= 1 - [P(X=3) + P(X=4)] \\&= 1 - \left[4C_3 \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^1 + 4C_4 \left(\frac{1}{2}\right)^4 \left(\frac{1}{2}\right)^0 \right] \\&= 1 - \left[\frac{4}{16} + \frac{1}{16} \right] \\&= \frac{11}{16}\end{aligned}$$

31. The overall percentage of passes in a certain examination is 80. If 6 candidates appear in the examination what is the probability that atleast 5 pass the examination

Solution :

$$\text{Let } p = \text{percentage of pass in examination} = \frac{80}{100} = \frac{4}{5}$$

$$q = 1 - p = \frac{1}{5}$$

$$n = 6$$

Let X denote the number of candidates of pass in examination.

X is the random variable follows a binomial distribution and

$$P(X=x) = nC_x p^x q^{n-x} \quad x = 0, 1, 2, \dots, 6$$

$$\begin{aligned}P(\text{Atleast 5 pass}) &= P(X \geq 5) \\&= P(X=5) + P(X=6) \\&= \left[6C_5 \left(\frac{4}{5}\right)^5 \left(\frac{1}{5}\right)^1 + 6C_6 \left(\frac{4}{5}\right)^6 \left(\frac{1}{5}\right)^0 \right]\end{aligned}$$

$$\begin{aligned}
 &= \frac{4^5}{5^6} [6+4] \\
 &= \frac{2048}{5^5} \\
 P(X \geq 5) &= \frac{2048}{3125}
 \end{aligned}$$

32. 20% of the bolts produced in a factory are found to be defective. Find the probability that in a sample of 10 bolts chosen at random exactly 2 will be defective using (i) Binomial distribution (ii) Poisson distribution. [$e^{-2} = 0.1353$].

Solution :

Given that $p = \frac{20}{100} = \frac{1}{5}$, $q = \frac{4}{5}$, $n = 10$

Let X denote the number of defective bolts chosen

(i) Using Binomial distribution

$$\begin{aligned}
 P(X=2) &= {}^{10}C_2 \left(\frac{1}{5}\right)^2 \left(\frac{4}{5}\right)^8 \\
 &= 45 \left(\frac{4^8}{5^{10}}\right)
 \end{aligned}$$

(ii) Using Poisson distribution

$$\lambda = np, = 10 \cdot \frac{1}{5} = 2$$

$$P(X=x) = \frac{e^{-\lambda} \lambda^x}{x!}$$

$$P(X=2) = \frac{e^{-2} 2^2}{2!}$$

$$\begin{aligned} P(x=2) &= 2 e^{-2} \\ &= 2 \times 0.1353 \\ &= 0.2706 \end{aligned}$$

33. Alpha particles are emitted by a radio active source at an average rate of 5 in a 20 minutes interval. Using Poisson distribution find the probability that there will be (i) 2 emission (ii) at least 2 emission in a particular 20 minutes interval. [$e^{-5} = 0.0067$].

Solution :

Average number of particles emitted in 20 minutes =5 i.e., $\lambda=5$

X be the number of particles emitted in 20 minutes

$$\begin{aligned} \text{(i)} P(X=2) &= e^{-5} \frac{5^2}{2!} \\ &= 0.0067 \times \frac{25}{2} \\ &= 0.0838 \end{aligned}$$

$$\begin{aligned} \text{(ii)} P(X \geq 2) &= 1 - P(X < 2) \\ &= 1 - [P(X=0) + P(X=1)] \\ &= 1 - e^{-5} \left[\frac{5^0}{0!} + \frac{5^1}{1!} \right] \end{aligned}$$

$$= 1 - 0.0067 \times 6$$

$$P(X \geq 2) = 0.9598$$

34. Suppose that the amount of cosmic radiation to which a person is exposed when flying by jet across the United States is a random variable having a normal distribution with a mean of 4.35 m rem and standard deviation of 0.59 m rem. What is the probability that a person will be exposed to more than 5.20 m rem of cosmic radiation of such flight

Solution :

Let X be the random variable normally distributed .

Given that $\mu = 4.35$, $\sigma = 0.59$

$$X = 5.20$$

$$Z = \frac{X - \mu}{\sigma} = \frac{5.2 - 4.35}{0.59}$$

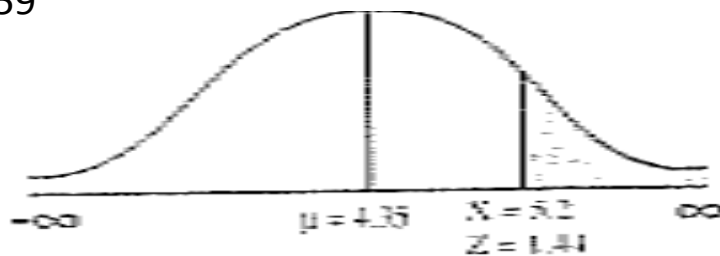
$$Z = \frac{0.85}{0.59} = 1.44$$

$$P(X > 5.20) = P(Z > 1.44) = P(1.44 < Z < \infty)$$

$$= 0.5 - P(0 < Z < 1.44)$$

$$= 0.5 - 0.4251$$

$$P(X > 5.20) = 0.0749$$



35. The life of army shoes is normally distributed with mean 8 months and standard deviation 2 months. If 5000 pairs are issued, how many pairs would be expected to need replacement within 12 months.

Solution :

Let x denote the life of army shoes.

$$\mu = 8, \sigma = 2, \quad N = 5000$$

$$X=12, Z = \frac{x-\mu}{\sigma} = \frac{12-8}{2} = 2$$

$$P(X \leq 12) = P(Z \leq 2)$$

$$= P(-\infty < Z < 2)$$

$$= P(-\infty < Z < 0) + P(0 < Z < 2)$$

$$= 0.5 + 0.4772$$

$$P(X \leq 12) = 0.9772$$

The probability for a shoe need to be replaced is 0.9772

Out of 5000 pairs of shoes number of pairs to need replacement

$$= 5000 \times 0.9772$$

$$= 4886 \text{ pair of shoes}$$

36. If the height of 300 students are normally distributed with mean 64.5 inches and standard deviation 3.3 inches, find the height below which 99% of the student lie.

Solution :

Let x denote the height of a student

$$\mu = 64.6 \text{ inches}, \sigma = 3.3 \text{ inches}$$

$$\text{Given that } P(-\infty < Z < c) = 0.99$$

$$P(-\infty < Z < 0) + P(0 < Z < c) = 0.99$$

$$0.5 + P(0 < Z < c) = 0.99$$

$$\therefore P(0 < Z < c) = 0.49 \quad \dots(1)$$

$$P(0 < Z < 2.33) = 0.49 \quad \dots(2)$$

From the table

$$\text{From (1) and (2)} \quad c = 2.33$$

$$Z = \frac{X - \mu}{\sigma}$$

$$2.33 = \frac{X - 64.5}{3.3}$$

$$X = 2.33 \times 3.33 + 64.5$$

$$X = 72.19 \text{ inches}$$

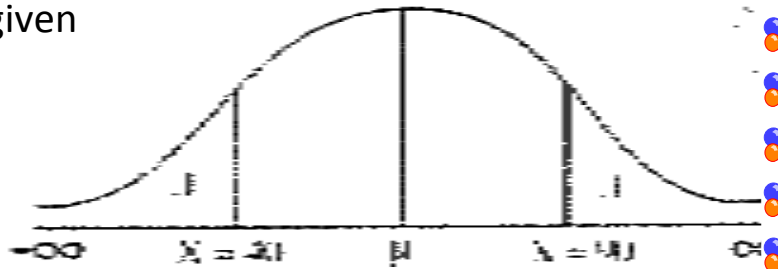
37. Marks in an aptitude test given to 800 students of a school was found to be normally distributed. 10% of the students scored below 40 marks and 10% of the students scored above 90 marks. Find the number of students scored between 40 and 90.

Solution :

Let x be the marks of a student given

$$P(X < 40) = 0.1$$

$$P(X > 90) = 0.1$$



$$\begin{aligned}
 P(40 < X < 90) &= P(-\infty < X < \infty) - \{ P(X < 40) + P(X > 90) \} \\
 &= 1 - 0.1 - 0.1 \\
 &= 0.8
 \end{aligned}$$

Out of 800 students number of student score between 40 and 90	}	$= 800 \times 0.8$ $= 640$ students
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